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Quantum-Dot Laser OCT

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Abstract

In this paper, we present the design and characterization of a monolithically integrated tunable laser for optical coherence tomography in medicine. This laser is the first monolithic photonic integrated circuit containing quantum-dot amplifiers, phase modulators, and passive components. We demonstrate electro-optical tuning capabilities over 60 nm between 1685 and 1745 nm, which is the largest tuning range demonstrated for an arrayed waveguide grating controlled tunable laser. Furthermore, it demonstrates that the active-passive integration technology designed for the 1550 nm telecom wavelength region can also be used in the 1600–1800 nm regions. The tunable laser has a 0.11 nm effective linewidth and an approximately 0.1 mW output power. Scanning capabilities of the laser are demonstrated in a free space Michelson interferometer setup where the laser is scanned over the 60 nm in 4000 steps with a 500 Hz scan frequency. Switching between two wavelengths within this 60 nm range is demonstrated to be possible within 500 ns .

Keywords- Integrated optoelectronics, laser tuning, optical imaging, quantum dot lasers.

I. INTRODUCTION

The designed InP-based laser system is based on a ring laser structure. The advantage of a ring laser above a linear laser with cleaved mirrors is that there is more freedom in design parameters like cavity length and output coupling. The ring basically consists of two 8 mm long intra cavity quantum-dot (QD) amplifiers [16] and two intra cavity tunable filters [17]. A schematic picture of the ring laser system is presented in Fig. 1. The special QD amplifiers are used to generate and amplify light in the 1700 nm wavelength region [15]. For the filters, tunable arrayed waveguide grating filters have been chosen that include electro-optic phase modulators. The two electro-optically tunable filters combined are used to select the wavelength in the laser cavity. Simulations on the laser [18] show that an intra cavity filter with a parabolic filter shape with a full width-half-maximum (FWHM) less than 0.5 nm and a free spectral range (FSR) larger than 200 nm should be sufficient to fulfill the laser requirements. For the design of this laser a combination of a high-resolution (HR) filter and a low-resolution (LR) filter is chosen to minimize the number of arms in both waveguide arrays. For the HR-filter an AWG type filter is used and for the LR-filter an MMI-tree type of filter is used. The LR-filter suppresses the transmission of the unwanted orders of transmission of the HR AWG filter. 5 mm long phase modulators (PHMs) are placed in the arms of both filters to make the filters tunable. The PHMs are voltage controlled and their low power consumption combined with the speed attainable are the main reasons for selecting this type of tunable filter. The combination of the two filters can be used to tune the filter combination over more than 200 nm within the 1600 nm to 1800 nm wavelength range. Further details and results on these filters are presented in [17]. The total ring length is approximately 43.5 mm. Part of the light in the ring cavity is coupled out with a 50% 2 × 2 multimode interference (MMI) coupler. Light coupled out from the clockwise (CLW) directional operation of the laser is feedback into counter clockwise (CCLW) directional operation of the laser with a MMI loop mirror [19]. The total length of the MMI loop mirror is designed to be 16.26 mm. This length was chosen to make that if there is an internal reflection in the ring cavity itself, the mode structure originating from that reflection and the feedback would be of the same order as the free spectral range of the ring. The light which is coupled out of the ring in the 2 × 2 MMI coupler originating from the CCLW direction is led through an 8 mm long QD output amplifier to boost the output signal. The high resolution tunable AWG filter [17] has monitor outputs which couple out a fraction of the light in the laser cavity due to the position of these waveguides on a higher order focal point of the free propagation regions (FPR). The output waveguide of the output amplifier and of the monitor outputs exit the chip under an angle of 7 degrees relative to the normal of the output facet to minimize facet reflections. All waveguides are low contrast shallow etched (100 nm into the film layer) waveguides to minimize waveguide losses. The complete mask layout of the laser is presented in Fig. 2. In the center of the mask the AWG-type HR-filter is located including the twenty-eight 5 mm-long PHMs. Directly below this HR-filter the MMI-tree-type LR-filter is located including 8 PHMs. The two 8 mm long QD ring amplifiers and the 8 mm long output amplifier are located below these two filters. On the mask the 170 μm wide, 8 mm long gold contacts on these amplifiers can clearly be seen. Each of the PHM in the filters is connected to one of the bond pads. Around the PHMs in the filters, 20 extra PHMs and waveguides are located to get a uniform structured area necessary for polyimide planarization as discussed in [17]. Five of these test PHMs are also connected to bond pads. Extra monitor outputs on the free propagation region (FPR) of the HR AWG filter are included in order to be able to calibrate the filter using ASE from a QD amplifier. On both FPRs of the HR filter, two of the monitor outputs are positioned at a higher diffraction order of the filter in order to monitor the light in the cavity during laser operation. The MMI loop mirror is located on the left hand side of the mask and is connected to one of the outputs of the MMI output coupler.

II. FABRICATION

The laser chip is fabricated within the generic active-passive integration technology used at COBRA [13], [14] using only shallow ridge waveguides. The devices were fabricated on wafers that contained active as well as passive areas to realize both active layerstack components (Semiconductor Optical Amplifiers (SOA)) as well as passive layerstack components (waveguides, AWGs, MMIs and PHMs). The active-passive layerstack has been fabricated using the butt-joint integration approach [20]. The active layerstack is first grown on an ntype InP (100) substrate by metal-organic vapor-phase epitaxy (MOVPE), as presented in [21]. In the active region, above the 500 nm n-InP buffer layer, five InAs quantum dot (QD) layers are stacked with an ultrathin GaAs interlayer underneath each QD layer to control the size of the QDs. These QD layers are placed in the center of a 500 nm InGaAsP not intentionally doped (n.i.d.) (Q.125) optical waveguiding core layer. The QD layers are designed to produce a gain spectrum in the 1600 nm to 1800 nm wavelength region [16]. The passive areas are selectively etched back till 20 nm underneath the QD layers. In the first regrowth step the passive InGaAsP (n.i.d. Q1.25) film layer is grown. In the second regrowth step the common 1.5 μm p-InP top cladding is grown followed by a compositionally graded 300 nm p-InGaAs(P) top contact layer. The devices are fabricated using a three-step CH4-H2 reactive-ion dry etch process to create shallow etched waveguides with or without contact layer and isolation section to prevent electrical crosstalk between PHMs and SOAs. The structures are planarized using six layers of polyimide. These six layers of polyimide are necessary to increase the surface flatness of the polyimide which in case is necessary to open all PHM and SOA at the same time prior to metal evaporation [17]. For this reason the PHMs are also equally spaced with a fixed 30 μm pitch to reduce non-uniform polyimide planarization. Height variations in the polyimide cause height variations in the opening of the PHM and SOAs. This leads either to polyimide in between part of the PHMs and the metal when the polyimide is not enough etched away or leads to areas where too much polyimide is etched away. This leads to higher waveguide losses due to the reduced spacing between the metal and the optical mode. Furthermore, using the six layers, a thicker total layer of polyimide has to be etched away which results in a rough surface. This roughness increases the adhesion of the metal to the polyimide. Evaporated Ti/Pt/Au metal pads contact the PHMs and SOAs to apply a voltage or a current. The SOA contact pads are thickened with plated Au to reduce the electrical resistance. The backside of the n-InP substrate is metalized to create a common ground contact. The structures are cleaved off from the rest of the wafer and no coating is applied to the facets. A schematic picture of the layerstack and the different components is depicted in Fig. 3. A picture of the 10 by 6 mm chip is depicted in Fig. 4.

III. LASER CHARACTERIZATION

The tunable laser system has been glued on a copper mount. The temperature of this oxygen free copper mount was controlled using water cooling to a temperature of 13° Celsius. The heat that is to be removed is solely generated in the quantum-dot amplifiers. More active temperature control is not necessary since the current injection in the amplifiers is kept constant and the reverse bias currents through the electrooptical phase modulators (PHM) of the tunable filters are several orders of magnitude lower. The PHMs are contacted via bond wires to a printed circuit board (PCB). On the PCB high bandwidth (1 GHz) multi-pin connectors, each connecting eight voltage signals, are positioned to connect the laser to the electronics. The PCB is also mounted on the copper chuck. This method avoids the use of multi-probes directly on the fragile chip. The current to the QD-amplifiers is provided via probe needles on the p-contact pads of the amplifier. The light emitted by the laser system is collected with a lensed fiber, either from a monitor output or from the output amplifier and analyzed with a spectrum analyzer with a 0.05 nm resolution (YOKOGAWA AQ6375). To be able to tune the laser in 1000 steps (0.1 nm steps) over 100 nm with a scan speed of 20 kHz the laser needs to be tuned within 50 ns. For the PHM in the arms of the filter this means a tuning should take place in less than 50 ns, preferably in the 1–10 ns range to leave some time for the selected laser mode to stabilize. The PHMs themselves have been demonstrated to be capable to switch in the 1–10 ns range [10]. More difficult is the control electronics which is required to apply the reverse bias voltage on all PHMs in parallel. In principle the voltage step size used to scan the filters with small wavelength steps (e.g. 0.1 nm for HR-filter) is in the order of 40 mV for the central PHMs and at most 0.6 V for the outermost PHM [17]. A large step in voltage is only necessary when a PHM setting needs to cross 2π rad and the phase is truncated to 0 rad. This means for the electrical control a step of approximately 4 V which requires a higher slew rate than the smaller voltage steps. These 2π steps fortunately occur for each PHM at a different time in the scan. If this large voltage step on one PHM is slightly slower than the other PHM, it has a minor influence on the switching speed. For the electrical control of these 38 PHM in the tunable filters, 100 MHz analog waveform generators that have been developed for this application are used [22]. These 13 bit resolution wave generators have a voltage range between −10 V and +10 V (we will only use the −10 V to 0 V range for the PHMs). A waveform pattern can be uploaded into a 4096 word memory for each wave generator. A common clock and common trigger signal can be used to step through each waveform pattern for the PHMs at the same time. The minimum step size is 10 ns and settling time of 4 ns between +0.5 V and −0.5 V (40 ns between +5 V and −5 V) at the output of the electronics. We expect a total optical loss inside the laser cavity of the passive components in the order of 25–30 dB. This means that relatively high current densities will be used in the amplifiers (4 kA/cm2 and above). This in turn means that the peak gain wavelength of the QD amplifiers will not shift much at and above threshold. We therefore start by looking at P-I curves from the laser with the filter set near the peak wavelength of the gain. The threshold of the laser has been measured by connecting both ring amplifiers together to a single current source. The optical output power is measured as a function of input current between 0 A and 3 A. The optical output power is measured on both monitor outputs of the laser to compare CLW and CCLW operation in the laser cavity. In Fig. 5 the P-I curve for the monitor outputs is given for the laser operation at 1715 nm, both filters are tuned to 1715 nm. From this P-I curve it can clearly be seen that the laser threshold is at 1500 mA. Above 2 A injection current the laser appears to start operating unidirectionally and switches between CLW or CCLW when the current increases [23]. Measurements at a number of different tuning wavelengths and at 2 A injection current show that the output power from the CLW direction monitor output was approximately 2.3 dB higher than the CCLW monitor output. From this observation it follows that the suppression of the CLW direction with the loop mirror is not visible in this current range. This results in an unpredictable switching between CLW and CCLW operation above 2 A injection current. The suppression of the CLW direction is expected to work in case of a higher feedback signal from CLW into CCLW direction with the loop mirror. This can either by reducing the losses in the feedback loop or by increasing the light intensity in the ring laser. This has not been further explored yet. P-I curves at other wavelengths show similar behavior with a threshold current between 1500 mA and 1750 mA. However unidirectional operation is not always observed. Because of the unpredictable operation direction above 2 A we chose to use a 1 A current through each ring amplifiers in the rest of the measurements presented in this work. This puts an upper limit on the gain in the amplifiers and therefore also the tuning ranges of the laser and the output power. When the output amplifier is used, a current of 700 mA is injected in the output amplifier. Higher currents through this amplifier did not increase the effective optical power in the laser peak, only the broadband amplified spontaneous emission (ASE).

A. Laser Tuning–Influence of the Gain Spectrum

At first the tuning behavior of the ring cavity has been studied by measuring the CCLW operation from the monitor output. Both ring amplifiers are biased at 1A forward current. For the tuning of the filters the calibration files of the filters [17] have been used (without any modification or optimization) to calculate the setting points of each PHM in the filters for each desired wavelength. The output spectrum is recorded with a 0.05 nm resolution spectrum analyzer. The laser output spectrum and power has been studied at tuning wavelengths between 1670 nm and 1770 nm in 1 nm steps around the approximate gain peak of the QD gain spectrum at 1715 nm. Between 1702 nm and 1733 nm we could clearly see a single laser peak with a detuning between +0.1 nm and +0.3 nm from the target wavelength and a FWHM less than 0.15 nm. An example of output spectrum at 1715 nm is depicted in Fig. 6 (black curve). Outside this 31 nm wavelength region there is on both sides of the spectrum a region (the regions 1695–1701 nm and 1734–1745 nm) where the laser wavelength sometimes jumped away from the set wavelength value. The laser started to operate close to the target wavelength or at a wavelength approximately 10 nm from the target wavelength (next passband of the high resolution filter). An example of this can be seen in Fig. 6 for the target wavelength 1742 nm (light gray curve), most output power is in a mode at 1732 nm. In some cases the laser started to work on two passband wavelengths of the HR filter, as can be seen in Fig. 6 for the target wavelength 1696 nm (dark gray curve). Outside the 1695 nm−1745 nm wavelength region the laser did not reach the lasing threshold. A closer look at the spectra showed that in the 1695–1701 nm region the laser tended to operate 10 nm towards the longer wavelengths and in the 1734–1745 nm region the laser tended to operate 10 nm towards the shorter wavelength region. This clearly indicated that the passband of the high resolution filter 10 nm towards the peak in the gain spectrum was not sufficiently suppressed by the low resolution filter. Outside the 1695–1745 nm wavelength region the ring laser did not reach the lasing threshold resulting in only ASE at the output. From the laser behavior presented above it can be concluded that the unwanted passbands of the high resolution filter are not sufficiently suppressed in wavelength regions where gain spectrum of the amplifiers is strongly wavelength dependent. In these initial measurements both filters were tuned to get the central wavelength of both filters at the target wavelength. This results in an equal suppression of both neighbor passbands, +/−10 nm from the desired passband of the HR-filter. The gain in the QD amplifiers can however have a strong slope over this 20 nm resulting in a larger roundtrip gain on the side of the peak wavelength of the QD amplifier in compare to the other side. A schematic representation of this situation is given in Fig. 7a. To compensate for this asymmetric gain profile around the target wavelength, the LR filter can be shifted a fraction away from the peak in the gain spectrum. A schematic representation of this situation is given in Fig. 7b. This increases the loss towards the peak in the gain spectrum, suppressing the unwanted lasing in the neighbor passband of the HRfilter, 10 nm towards the gain peak. Optimizing the laser output by a simple detuning of the MMI filter as indicated in Fig. 7b did not improve the suppression of the lasing in unwanted modes in all cases. Nearly complete suppression of these modes could however be obtained by using a kind of recalibration procedure of the LR-filter. In this procedure the individual PHMs of the LR-filter are scanned over the voltage range necessary to reach 0 to 2π phase shifting. During this scan, the suppression of the two laser peaks at 10 nm from the target wavelength is measured relative to the laser peak at the target wavelength. The voltage at which the suppression is the highest is stored and directly applied for each scanned PHM. The extracted voltage array containing the voltages for each PHM to get the highest suppression of the neighbor pass-bands of the HR-filter for one wavelength is stored. This procedure was executed each 5 nm over the tuning range of the laser. After the execution of the procedure over the complete spectrum, the phase settings for intermediate wavelengths could be determined by interpolation. Using the new calibration data of the LR-filter the spectral higher order mode suppression has been measured between 1670 nm and 1750 nm in 1 nm steps. The results for the mode obtained in the 1685 to 1745 nm wavelength range the suppression in most cases is better than 25 dB. Only around 1727 nm we could still observe a second laser peak around 1717 in the laser output spectrum. In the measurements this unwanted laser peak was still 15 dB lower than the peak at the target wavelength 1727 nm.

B. *Laser Tuning–Influence of the Cavity Mode Structure*

An extensive characterization of the laser performance has been executed over the complete tuning range of 1685 nm to 1745 nm. The laser is tuned over this range in 4000 steps (0.015 nm steps). An overview of the measurement results is presented in Fig. 9. Except from a small wavelength region between 1726 nm and 1727 nm the laser system was lasing between 1686 nm and 1745 nm (Fig. 9a/b). The detuning with respect to the target wavelength was in all cases between −0.2 nm and +0.2 nm (Fig. 9c) and the FWHM of the laser peak between 0.05 nm and 0.30 nm (Fig. 9d). Between 1726 nm and 1727 nm the lasing wavelength could so far not be guaranteed to work only on the target wavelength. In this small range a second laser peak appears at 10 nm from the target wavelength resulting in dual wavelength operation. Remarkable in the measurements presented above are the 0.1 nm variation band in the detuning, the large 0.25 nm fluctuation in the FWHM and the fluctuation in the measured peak power. A closer look at the output spectrum when the laser is tuned over 1 nm bandwidth gave more information about the origin of these fluctuations. In Fig. 10a the peak wavelength of the laser peak is presented with respect to the target wavelength. Its seen that the peak wavelength jumps with approximately 0.1 nm steps through the spectrum while tuning the laser with constant steps. The spectra of a series of measurements between 1715.420 nm and 1716.00 are given in Fig. 10b. Also from these spectra it can clearly be seen that the ring cavity has a preferred operating wavelength at 1715.5 nm and 1715.6 nm. Tuning the laser in between these two wavelengths results in a combination of these two preferred wavelengths. The FWHM will thus be higher and the peak power will be proportionally lower. This behavior explains the fluctuations in Fig. 9. These 0.1 nm spectral jumps indicate a cavity within the ring structure or some feedback into the cavity. A laser cavity with an 8.1 mm long roundtrip results in an extra mode structure with a 0.1 nm spacing. The most probable locations from which reflections in the ring laser system can be expected are the isolation sections at both sides of the amplifiers and PHMs. A possible explanation of the 0.1 nm mode structure is that a combination of a reflection on both sides of the 8 mm long amplifiers, resulting in a 0.05 nm mode spacing, in combination with the 0.02 nm mode spacing from the 43.5 mm long ring cavity. A combination of this 0.05 nm and 0.02 nm could result in the 0.1 nm mode structure. This has however not yet been explored.

C. Laser Tuning Speed

As an indication of the attainable tuning speed of the laser, the switching behavior of the laser has been measured between two wavelengths separated by tens of nanometers. When the laser is scanned using small wavelength steps (<0.1 nm) it can in principle switch faster than when switching between two widely separated wavelengths. This is due to the fact that there is a higher light intensity in cavity modes close to the lasing wavelength compared to those a couple of nm away from the laser peak. This reduces the build-up time for the lasing close to the original laser peak. The tuning speed over a very small wavelength region could not be directly measured; the measurement we performed gives a lower limit on the attainable scanning speed. To measure the switching over a couple of nm, the output from the laser is passed through a free space interference band-pass filter, filtering out one of the wavelengths and passing through the other wavelength. The light through the filter is collected on an amplified InGaAs photodiode with a rise time of 80 ns (3 MHz bandwidth) and then recorded using a 1GHz bandwidth oscilloscope. The laser is switched at a 1 kHz repetition rate between 1700 nm and 1745 nm with the band-pass filter at 1700 nm. The 10%–90% rise- and fall-times are measured to be 490 ns and 380 ns respectively. This is mainly the time the laser needs to build up the laser peak at the new wavelength. The switching time of the HR intra-cavity filter is approximately 100 ns [17] and t is limited mainly by the speed of the electrical control of the phase modulators. Therefore the influence on the switching speed of the filter on the measured switching time of the laser is limited. The difference between the expected 50 ns switching [18] and the measured 490 ns can be attributed to the fact the laser could not be operated further above its threshold. The unsaturated gain of the amplifiers is only just above total round trip loss which in turn leads to a long build-up time.

IV. LASER COHERENCE LENGTH AND EFFECTIVE LINEWIDTH

The performance of the tunable laser with respect to its use as a source for OCT has been studied using a free space Michelson interferometer setup. This is a first step towards using the tunable laser in an OCT system and can be used to determine the effective linewidth of the laser. A schematic picture of the free space Michelson interferometer setup is given in Fig. 11. The tunable laser, controlled by the control electronics, is used to scan the laser over 60 nm in 4000 steps with a 500 Hz scan rate in order to have 500 ns per step. The light from the laser is collected with a lensed fiber and coupled into the free space Michelson interferometer setup with a microscope objective. In the cubical beam splitter the light is equally separated in two arms, one towards a fixed mirror and the other to a movable mirror. After reflection on the two mirrors the light is again combined and collected with a microscope objective into a single mode fiber. The collected light is measured with a p-doped InGaAs detector with a 60 MHz bandwidth and traced on a digital oscilloscope (8 bit resolution) with a 10 ns time between samples. This 10 ns sampling results in 50 samples per wavelength. For each wavelength an average over the 30 central samples (nos. 15–45) is used to reduce the influence of the switching dynamics of the laser during tuning of the filters. The start of the laser scan is indicated on the oscilloscope with a trigger signal from the control electronics. The recorded trace on the oscilloscope contains the information on the reflection on the moving mirror with respect to the fixed mirror. The difference in distance of the two mirrors with respect to the beam splitter results in a modulation in the spectral domain. In Fig. 12a the recorded trace of one spectral scan is given in which the moving mirror is located 1 mm further from the beam splitter than the fixed mirror. The information in the trace has been extracted with an Inverse Nonlinear Extended Discrete Fourier Transform (INEDFT) [24]. In this INEDFT the spectral trace is Fourier transformed using the previously measured wavelengths at each wavelength step. The instable wavelength region between 1725 nm and 1727 nm is excluded from the Fourier transformation. The final time trace is an average over 20 calculated time traces. The time trace is transformed to a spatial trace using the speed of light. The spatial scale is divided by two to take into account the double path length between mirror and beam splitter. The resulting spatial trace is given in Fig. 3b. The peak at 1 mm originates from the reflection on the moving mirror. The x-axis represents the location of the reflection with respect to the location of the fixed mirror. The effective linewidth of the tunable laser over the complete wavelength tuning range can be determined using the free space Michelson interferometer setup. The peak intensity of the reflected signal is dependent on the relative path length difference between the reflection and the fixed reference mirror. This exponential intensity decay R, can be described as a function of imaging depth z [25] where *d* is the maximum scan depth $d = \lambda 2/(4\_\lambda)$ in which $\_\lambda$ is the wavelength sampling interval in the scan (in this case 0.015 nm) and ω is the ratio of the spectral resolution (FWHM) $\delta\lambda$ to the wavelength sampling interval $\omega = \delta\lambda/\_\lambda$. The spectral resolution can also be called as the effective linewidth of the tunable laser over the complete wavelength tuning range. To determine this effective linewidth a series of 29 measurements has been performed in which the relative path length difference of the moving mirror is stepwise increased with 0.25 mm. For each path length the spectrum is recorded on the oscilloscope over 20 scans. The recorded spectra are translated to the spatial domain with the INEDFT and averaged in the spatial domain over the 20 scans. The exponential decay function (1) is fitted on the peak intensities of the reflections. The spatial domain traces as well as the fitted curve are presented in Fig. 13. The effective linewidth $\delta\lambda$ of the tunable laser is determined to be 0.11 nm which corresponds to the average measured FWHM of the laser peak presented in Fig. 9d (an average over all 4000 measured FWHM gave 0.11 nm).The axial resolution of an OCT image δL is determined by the scan range $\_\lambda span$ = 60 nm of laser and can be calculated with:

where $\lambda 0$ = 1715 nm is the central wavelength in the wavelength span. The axial resolution for this tunable laser in an OCT system is 21.6 μm in vacuum.

We consider the following anycast field equations defined over an open bounded piece of network and /or feature space $\Omega \subset R^d$. They describe the dynamics of the mean anycast of each of p node populations.

$$\begin{cases} (\frac{d}{dt}+l_i)V_i(t,r)=\sum_{j=1}^{p}\int_\Omega J_{ij}(r,\bar{r})S[(V_j(t-\tau_{ij}(r,\bar{r}),\bar{r})-h_{|j})]d\bar{r} \\ \quad +I_i^{ext}(r,t), \quad t\geq 0, 1\leq i\leq p, \\ V_i(t,r)=\phi_i(t,r) \quad t\in[-T,0] \end{cases} \quad (1)$$

We give an interpretation of the various parameters and functions that appear in (1), Ω is finite piece of nodes and/or feature space and is represented as an open bounded set of R^d. The vector r and $\bar{r}$ represent points in Ω. The function $S:R\rightarrow(0,1)$ is the normalized sigmoid function:

$$S(z)=\frac{1}{1+e^{-z}} \quad (2)$$

It describes the relation between the input rate v_i of population i as a function of the packets potential, for example, $V_i=v_i=S[\sigma_i(V_i-h_i)]$. We note V the $p-$ dimensional vector $(V_1,...,V_p)$. The p function $\phi_i, i=1,...,p$, represent the initial conditions, see below. We note ϕ the $p-$ dimensional vector $(\phi_1,...,\phi_p)$. The p function $I_i^{ext}, i=1,...,p$, represent external factors from other network areas. We note I^{ext} the $p-$ dimensional vector $(I_1^{ext},...,I_p^{ext})$. The $p\times p$ matrix of functions $J=\{J_{ij}\}_{i,j=1,...,p}$ represents the connectivity between populations i and j, see below. The p real values $h_i, i=1,...,p$, determine the threshold of activity for each population, that is, the value of the nodes potential corresponding to 50% of the maximal activity. The p real positive values $\sigma_i, i=1,...,p$, determine the slopes of the sigmoids at the origin. Finally the p real positive values $l_i, i=1,...,p$, determine the speed at which each anycast node potential decreases exponentially toward its real value. We also introduce the function $S:R^p\rightarrow R^p$, defined by $S(x)=[S(\sigma_1(x_1-h_1)),...,S(\sigma_p-h_p))]$, and the diagonal $p\times p$ matrix $L_0=diag(l_1,...,l_p)$. Is the intrinsic dynamics of the population given by the linear response of data transfer. $(\frac{d}{dt}+l_i)$ is replaced by $(\frac{d}{dt}+l_i)^2$ to use the alpha function response. We use $(\frac{d}{dt}+l_i)$ for simplicity although our analysis applies to more general intrinsic dynamics. For the sake, of generality, the propagation delays are not assumed to be identical for all populations, hence they are described by a matrix $\tau(r,\bar{r})$ whose element $\tau_{ij}(r,\bar{r})$ is the propagation delay between population j at $\bar{r}$ and population i at r. The reason for this assumption is that it is still unclear from anycast if propagation delays are independent of the populations. We assume for technical reasons that τ is continuous, that is $\tau\in C^0(\bar{\Omega}^2,R_+^{p\times p})$. Moreover packet data indicate that τ is not a symmetric function i.e., $\tau_{ij}(r,\bar{r})\neq\tau_{ij}(\bar{r},r)$, thus no assumption is made about this symmetry unless otherwise stated. In order to compute the righthand side of (1), we need to know the node potential factor V on interval $[-T,0]$. The value of T is obtained by considering the maximal delay:

$$\tau_m=\max_{i,j(r,\bar{r}\in\bar{\Omega}\times\bar{\Omega})}\tau_{i,j}(r,\bar{r}) \quad (3)$$

Hence we choose $T=\tau_m$

A. Mathematical Framework

A convenient functional setting for the non-delayed packet field equations is to use the space $F=L^2(\Omega,R^p)$ which is a Hilbert space endowed with the usual inner product:

$$\langle V,U\rangle_F=\sum_{i=1}^{p}\int_\Omega V_i(r)U_i(r)dr \quad (1)$$

To give a meaning to (1), we defined the history space $C=C^0([-\tau_m,0],F)$ with $\|\phi\|=\sup_{t\in[-\tau_m,0]}\|\phi(t)\|F$, which is the Banach phase space associated with equation (3). Using the notation $V_t(\theta)=V(t+\theta),\theta\in[-\tau_m,0]$, we write (1) as

$$\begin{cases} \dot{V}(t)=-L_0V(t)+L_1S(V_t)+I^{ext}(t), \\ V_0=\phi\in C, \end{cases} \qquad (2)$$

Where

$$\begin{cases} L_1:C\rightarrow F, \\ \phi\rightarrow\int_\Omega J(.,\bar{r})\phi(\bar{r},-\tau(.,\bar{r}))d\bar{r} \end{cases}$$

Is the linear continuous operator satisfying $\|L_1\|\le\|J\|_{L^2(\Omega^2,R^{p\times p})}$. Notice that most of the papers on this subject assume Ω infinite, hence requiring $\tau_m=\infty$.

Proposition 1.0 If the following assumptions are satisfied.

1. $J\in L^2(\Omega^2,R^{p\times p})$,
2. The external current $I^{ext}\in C^0(R,F)$,
3. $\tau\in C^0(\overline{\Omega^2},R_+^{p\times p}),\sup_{\overline{\Omega^2}}\tau\le\tau_m$.

Then for any $\phi\in C$, there exists a unique solution $V\in C^1([0,\infty),F)\cap C^0([-\tau_m,\infty,F)$ to (3)

Notice that this result gives existence on R_+, finite-time explosion is impossible for this delayed differential equation. Nevertheless, a particular solution could grow indefinitely, we now prove that this cannot happen.

B. *Boundedness of Solutions*

A valid model of neural networks should only feature bounded packet node potentials.

Theorem 1.0 All the trajectories are ultimately bounded by the same constant R if $I\equiv\max_{t\in R^+}\|I^{ext}(t)\|_F<\infty$.

Proof :Let us defined $f:R\times C\rightarrow R^+$ as

$$f(t,V_t)\overset{def}{=}\langle -L_0V_t(0)+L_1S(V_t)+I^{ext}(t),V(t)\rangle_F=\frac{1}{2}\frac{d\|V\|_F^2}{dt}$$

We note $l=\min_{i=1,\dots p}l_i$

$$f(t,V_t)\le -l\|V(t)\|_F^2+(\sqrt{p|\Omega|}\|J\|_F+I)\|V(t)\|_F$$

Thus, if

$$\|V(t)\|_F\ge 2\frac{\sqrt{p|\Omega|}.\|J\|_F+I}{l}\overset{def}{=}R,\ f(t,V_t)\le-\frac{lR^2}{2}\overset{def}{=}-\delta<0$$

Let us show that the open route of F of center 0 and radius R,B_R, is stable under the dynamics of equation. We know that $V(t)$ is defined for all $t\ge 0s$ and that $f<0$ on ∂B_R, the boundary of B_R. We consider three cases for the initial condition V_0. If $\|V_0\|_C<R$ and set $T=\sup\{t\,|\,\forall s\in[0,t],V(s)\in\overline{B_R}\}$. Suppose that $T\in R$, then $V(T)$ is defined and belongs to $\overline{B_R}$, the closure of B_R, because $\overline{B_R}$ is closed, in effect to ∂B_R, we also have $\frac{d}{dt}\|V\|_F^2\,|_{t=T}=f(T,V_T)\le-\delta<0$ because $V(T)\in\partial B_R$. Thus we deduce that for $\varepsilon>0$ and small enough, $V(T+\varepsilon)\in\overline{B_R}$ which contradicts the definition of T. Thus $T\notin R$ and $\overline{B_R}$ is stable.

Because f<0 on $\partial B_R,V(0)\in\partial B_R$ implies that $\forall t>0,V(t)\in B_R$. Finally we consider the case $V(0)\in C\overline{B_R}$. Suppose that $\forall t>0,V(t)\notin\overline{B_R}$, then $\forall t>0,\frac{d}{dt}\|V\|_F^2\le-2\delta$, thus $\|V(t)\|_F$ is monotonically decreasing and reaches the value of R in finite time when $V(t)$ reaches ∂B_R. This contradicts our assumption. Thus $\exists T>0\,|\,V(T)\in B_R$.

Proposition 1.1 : Let s and t be measured simple functions on X. for $E\varepsilon M$, define

$$\phi(E)=\int_E s\,d\mu \qquad (1)$$

Then ϕ is a measure on M.

$$\int_X(s+t)d\mu=\int_X s\,d\mu+\int_X t\,d\mu \qquad (2)$$

Proof : If s and if $E_1,E_2,\dots$ are disjoint members of M whose union is E, the countable additivity of μ shows that

$$\phi(E)=\sum_{i=1}^n\alpha_i\mu(A_i\cap E)=\sum_{i=1}^n\alpha_i\sum_{r=1}^\infty\mu(A_i\cap E_r)$$

$$=\sum_{r=1}^\infty\sum_{i=1}^n\alpha_i\mu(A_i\cap E_r)=\sum_{r=1}^\infty\phi(E_r)$$

Also, $\varphi(\phi)=0,$ so that φ is not identically ∞.
Next, let s be as before, let $\beta_1,...,\beta_m$ be the distinct values of t,and let $B_j=\{x:t(x)=\beta_j\}$ If $E_{ij}=A_i\cap B_j,$ the

$$\int_{E_{ij}}(s+t)d\mu=(\alpha_i+\beta_j)\mu(E_{ij})$$

and $\int_{E_{ij}}sd\mu+\int_{E_{ij}}td\mu=\alpha_i\mu(E_{ij})+\beta_j\mu(E_{ij})$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1\le i\le n, 1\le j\le m)$, the first half of our proposition implies that (2) holds.

Theorem 1.1: If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of , and if $\varepsilon>0$, then there exists a polynomial P such that $|f(z)=P(z)|<\varepsilon$ for all $z\varepsilon K$. If the interior of K is empty, then part of the hypothesis is vacuously satisfied, and the conclusion holds for every $f\varepsilon C(K)$. Note that K need to be connected.

Proof: By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension and denote it again by f. For any $\delta>0$, let $\omega(\delta)$ be the supremum of the numbers $|f(z_2)-f(z_1)|$ Where z_1 and z_2 are subject to the condition $|z_2-z_1|\le\delta$. Since f is uniformly continous, we have $\lim_{\delta\to 0}\omega(\delta)=0$ (1) From now on, δ will be fixed. We shall prove that there is a polynomial P such that

$$|f(z)-P(z)|<10{,}000\ \omega(\delta)\quad (z\varepsilon K)\qquad (2)$$

By (1), this proves the theorem. Our first objective is the construction of a function $\Phi\varepsilon C_c^{'}(R^2)$, such that for all z

$$|f(z)-\Phi(z)|\le\omega(\delta),\qquad (3)$$

$$|(\partial\Phi)(z)|<\frac{2\omega(\delta)}{\delta},\qquad (4)$$

And

$$\Phi(z)=-\frac{1}{\pi}\iint_X\frac{(\partial\Phi)(\zeta)}{\zeta-z}d\zeta d\eta\qquad(\zeta=\xi+i\eta),\qquad (5)$$

Where X is the set of all points in the support of Φ whose distance from the complement of K does not δ. (Thus X contains no point which is "far within" K.) We construct Φ as the convolution of f with a smoothing function A. Put $a(r)=0$ if $r>\delta,$ put

$$a(r)=\frac{3}{\pi\delta^2}(1-\frac{r^2}{\delta^2})^2\qquad(0\le r\le\delta),\qquad (6)$$

And define

$$A(z)=a(|z|)\qquad (7)$$

For all complex z. It is clear that $A\varepsilon C_c^{'}(R^2)$. We claim that

$$\iint_{R^s}A=1,\qquad (8)$$

$$\iint_{R^2}\partial A=0,\qquad (9)$$

$$\iint_{R^3}|\partial A|=\frac{24}{15\delta}<\frac{2}{\delta},\qquad (10)$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because A has compact support. To compute (10), express ∂A in polar coordinates, and note that $\partial A/\partial\theta=0,$

$$\partial A/\partial r=-a',$$

Now define

$$\Phi(z)=\iint_{R^2}f(z-\zeta)Ad\xi d\eta=\iint_{R^2}A(z-\zeta)f(\zeta)d\xi d\eta\qquad (11)$$

Since f and A have compact support, so does Φ. Since

$$\Phi(z)-f(z)$$
$$=\iint_{R^2}[f(z-\zeta)-f(z)]A(\xi)d\xi d\eta\quad (12)$$

And $A(\zeta)=0$ if $|\zeta|>\delta,$ (3) follows from (8). The difference quotients of A converge boundedly to the corresponding partial derivatives, since $A\varepsilon C_c^{'}(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$(\partial\Phi)(z)=\iint_{R^2}(\overline{\partial A})(z-\zeta)f(\zeta)d\xi d\eta$$
$$=\iint_{R^2}f(z-\zeta)(\partial A)(\zeta)d\xi d\eta$$
$$=\iint_{R^2}[f(z-\zeta)-f(z)](\partial A)(\zeta)d\xi d\eta \qquad (13)$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\partial\Phi$, we see that Φ has continuous partial derivatives, if we can show that $\partial\Phi=0$ in G, where G is the set of all $z\varepsilon K$ whose distance from the complement of K exceeds δ. We shall do this by showing that

$$\Phi(z)=f(z) \qquad (z\varepsilon G); \qquad (14)$$

Note that $\partial f=0$ in G, since f is holomorphic there. Now if $z\varepsilon G$, then $z-\zeta$ is in the interior of K for all ζ with $|\zeta|<\delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\Phi(z)=\int_0^\delta a(r)rdr\int_0^{2\pi}f(z-re^{i\theta})d\theta$$
$$=2\pi f(z)\int_0^\delta a(r)rdr=f(z)\iint_{R^2}A=f(z) \qquad (15)$$

For all $z\ \varepsilon\ G$, we have now proved (3), (4), and (5) The definition of X shows that X is compact and that X can be covered by finitely many open discs $D_1,\dots,D_n$, of radius 2δ, whose centers are not in K. Since S^2-K is connected, the center of each D_j can be joined to ∞ by a polygonal path in S^2-K. It follows that each D_j contains a compact connected set E_j, of diameter at least 2δ, so that S^2-E_j is connected and so that $K\cap E_j=\phi$. with $r=2\delta$. There are functions $g_j\varepsilon H(S^2-E_j)$ and constants b_j so that the inequalities.

$$\left|Q_j(\zeta,z)\right|<\frac{50}{\delta}, \qquad (16)$$

$$\left|Q_j(\zeta,z)-\frac{1}{z-\zeta}\right|<\frac{4{,}000\delta^2}{|z-\zeta|^2} \qquad (17)$$

Hold for $z\notin E_j$ and $\zeta\in D_j$, if

$$Q_j(\zeta,z)=g_j(z)+(\zeta-b_j)g_j^2(z) \qquad (18)$$

Let Ω be the complement of $E_1\cup\dots\cup E_n$. Then Ω is an open set which contains K. Put $X_1=X\cap D_1$ and $X_j=(X\cap D_j)-(X_1\cup\dots\cup X_{j-1})$, for $2\le j\le n$,

Define

$$R(\zeta,z)=Q_j(\zeta,z) \qquad (\zeta\varepsilon X_j, z\,\varepsilon\,\Omega) \qquad (19)$$

And

$$F(z)=\frac{1}{\pi}\iint_X(\partial\Phi)(\zeta)R(\zeta,z)d\zeta d\eta \qquad (20)$$
$$(z\ \varepsilon\ \Omega)$$

Since,

$$F(z)=\sum_{j=1}\frac{1}{\pi}\iint_{X_i}(\partial\Phi)(\zeta)Q_j(\zeta,z)d\xi d\eta, \qquad (21)$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2. Hence $F\varepsilon H(\Omega)$. By (20), (4), and (5) we have

$$|F(z)-\Phi(z)|<\frac{2\omega(\delta)}{\pi\delta}\iint_X|R(\zeta,z)-\frac{1}{z-\zeta}|d\xi d\eta \quad (z\ \varepsilon\ \Omega) \quad (22)$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta\ \varepsilon\ X$ and $z\ \varepsilon\ \Omega$. Now fix $z\ \varepsilon\ \Omega$., put $\zeta=z+\rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho<4\delta$, by (17) if $4\delta\le\rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi\int_0^{4\delta}\left(\frac{50}{\delta}+\frac{1}{\rho}\right)\rho d\rho=808\pi\delta \qquad (23)$$

And

$$2\pi\int_{4\delta}^{\infty}\frac{4{,}000\delta^2}{\rho^2}\rho d\rho=2{,}000\pi\delta. \qquad (24)$$

Hence (22) yields

$$|F(z)-\Phi(z)|<6{,}000\omega(\delta) \qquad (z\ \varepsilon\ \Omega) \qquad (25)$$

Since $F\ \varepsilon\ H(\Omega)$, $K\subset\Omega$, and S^2-K is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0 : Suppose $f\varepsilon C_c^{'}(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad (1)$$

Then the following "Cauchy formula" holds:

$$f(z) = -\frac{1}{\pi}\iint_{R^2}\frac{(\partial f)(\zeta)}{\zeta - z}d\xi d\eta$$

$$(\zeta = \xi + i\eta) \qquad (2)$$

Proof: This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r,\theta) = f(z + re^{i\theta}),\ r > 0,\ \theta$ real

If $\zeta = z + re^{i\theta}$, the chain rule gives

$$(\partial f)(\zeta) = \frac{1}{2}e^{i\theta}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right]\varphi(r,\theta) \qquad (3)$$

The right side of (2) is therefore equal to the limit, as $\varepsilon \to 0$, of

$$-\frac{1}{2}\int_{\varepsilon}^{\infty}\int_{0}^{2\pi}\left(\frac{\partial \varphi}{\partial r} + \frac{i}{r}\frac{\partial \varphi}{\partial \theta}\right)d\theta dr \qquad (4)$$

For each $r > 0, \varphi$ is periodic in θ, with period 2π. The integral of $\partial\varphi / \partial\theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_{\varepsilon}^{\infty}\frac{\partial \varphi}{\partial r}dr = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\varepsilon,\theta)d\theta \qquad (5)$$

As $\varepsilon \to 0,\ \varphi(\varepsilon,\theta) \to f(z)$ uniformly. This gives (2)

If $X^{\alpha} \in a$ and $X^{\beta} \in k[X_1,...X_n]$, then $X^{\alpha}X^{\beta} = X^{\alpha+\beta} \in a$, and so A satisfies the condition $(*)$. Conversely,

$$(\sum_{\alpha \in A} c_{\alpha}X^{\alpha})(\sum_{\beta \in \square^n} d_{\beta}X^{\beta}) = \sum_{\alpha,\beta} c_{\alpha}d_{\beta}X^{\alpha+\beta} \qquad (\textit{finite sums}),$$

and so if A satisfies $(*)$, then the subspace generated by the monomials $X^{\alpha}, \alpha \in a$, is an ideal. The proposition gives a classification of the monomial ideals in $k[X_1,...X_n]$: they are in one to one correspondence with the subsets A of $\square^n$ satisfying $(*)$. For example, the monomial ideals in $k[X]$ are exactly the ideals $(X^n),\ n \geq 1$, and the zero ideal (corresponding to the empty set A). We write $\langle X^{\alpha} \mid \alpha \in A\rangle$ for the ideal corresponding to A (subspace generated by the $X^{\alpha}, \alpha \in a$).

LEMMA 1.1. Let S be a subset of $\square^n$. The the ideal a generated by $X^{\alpha}, \alpha \in S$ is the monomial ideal corresponding to

$$A \overset{df}{=} \{\beta \in \square^n \mid \beta - \alpha \in \square^n, \quad some\ \alpha \in S\}$$

Thus, a monomial is in a if and only if it is divisible by one of the $X^{\alpha}, \alpha \in| S$

PROOF. Clearly A satisfies $(*)$, and $a \subset \langle X^{\beta} \mid \beta \in A\rangle$. Conversely, if $\beta \in A$, then $\beta - \alpha \in \square^n$ for some $\alpha \in S$, and $X^{\beta} = X^{\alpha}X^{\beta-\alpha} \in a$. The last statement follows from the fact that $X^{\alpha} \mid X^{\beta} \Leftrightarrow \beta - \alpha \in \square^n$. Let $A \subset \square^n$ satisfy $(*)$. From the geometry of A, it is clear that there is a finite set of elements $S = \{\alpha_1,...\alpha_s\}$ of A such that $A = \{\beta \in \square^n \mid \beta - \alpha_i \in \square^2,\ some\ \alpha_i \in S\}$ (The α_i's are the corners of A) Moreover, $a \overset{df}{=} \langle X^{\alpha} \mid \alpha \in A\rangle$ is generated by the monomials $X^{\alpha_i}, \alpha_i \in S$.

DEFINITION 1.0. For a nonzero ideal a in $k[X_1,...,X_n]$, we let $(LT(a))$ be the ideal generated by

$$\{LT(f) \mid f \in a\}$$

LEMMA 1.2 Let a be a nonzero ideal in $k[X_1,...,X_n]$; then $(LT(a))$ is a monomial ideal, and it equals $(LT(g_1),...,LT(g_n))$ for some $g_1,...,g_n \in a$.

PROOF. Since $(LT(a))$ can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of a.

THEOREM 1.2. Every *ideal* a in $k[X_1,...,X_n]$ is finitely generated; more precisely, $a = (g_1,...,g_s)$ where $g_1,...,g_s$ are any elements of a whose leading terms generate $LT(a)$

PROOF. Let $f \in a$. On applying the division algorithm, we find $f = a_1 g_1 + ... + a_s g_s + r, \quad a_i, r \in k[X_1, ..., X_n]$, where either $r = 0$ or no monomial occurring in it is divisible by any $LT(g_i)$. But $r = f - \sum a_i g_i \in a$, and therefore $LT(r) \in LT(a) = (LT(g_1), ..., LT(g_s))$, implies that every monomial occurring in r is divisible by one in $LT(g_i)$. Thus $r = 0$, and $g \in (g_1, ..., g_s)$.

DEFINITION 1.1. A finite subset $S = \{g_1, | ..., g_s\}$ of an ideal a is a standard ($(Gr\ddot{o}bner)$ bases for a if $(LT(g_1), ..., LT(g_s)) = LT(a)$. In other words, S is a standard basis if the leading term of every element of a is divisible by at least one of the leading terms of the g_i.

THEOREM 1.3 *The ring $k[X_1, ..., X_n]$ is Noetherian i.e., every ideal is finitely generated.*

PROOF. For $n = 1$, $k[X]$ is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on n. Note that the obvious map $k[X_1, ... X_{n-1}][X_n] \to k[X_1, ... X_n]$ is an isomorphism – this simply says that every polynomial f in n variables $X_1, ... X_n$ can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1, ..., X_n]$:

$$f(X_1, ... X_n) = a_0(X_1, ... X_{n-1}) X_n^r + ... + a_r(X_1, ... X_{n-1})$$

Thus the next lemma will complete the proof

LEMMA 1.3. If A is Noetherian, then so also is $A[X]$

PROOF. For a polynomial

$$f(X) = a_0 X^r + a_1 X^{r-1} + ... + a_r, \quad a_i \in A, \quad a_0 \neq 0,$$

r is called the degree of f, and a_0 is its leading coefficient. We call 0 the leading coefficient of the polynomial 0. Let a be an ideal in $A[X]$. The leading coefficients of the polynomials in a form an ideal a' in A, and since A is Noetherian, a' will be finitely generated. Let $g_1, ..., g_m$ be elements of a whose leading coefficients generate a', and let r be the maximum degree of g_i. Now let $f \in a$, and suppose f has degree $s > r$, say, $f = aX^s + ...$ Then $a \in a'$, and so we can write $a = \sum b_i a_i, \quad b_i \in A,$

$a_i = leading\ coefficient\ of\ g_i$

Now

$f - \sum b_i g_i X^{s - r_i}, \quad r_i = \deg(g_i)$, has degree $< \deg(f)$. By continuing in this way, we find that $f \equiv f_t \quad \mod(g_1, ... g_m)$ With f_t a polynomial of degree $t < r$. For each $d < r$, let a_d be the subset of A consisting of 0 and the leading coefficients of all polynomials in a of degree d; it is again an ideal in A. Let $g_{d,1}, ..., g_{d,m_d}$ be polynomials of degree d whose leading coefficients generate a_d. Then the same argument as above shows that any polynomial f_d in a of degree d can be written $f_d \equiv f_{d-1} \quad \mod(g_{d,1}, ... g_{d,m_d})$ With f_{d-1} of degree $\leq d - 1$. On applying this remark repeatedly we find that $f_t \in (g_{r-1,1}, ... g_{r-1,m_{r-1}}, ... g_{0,1}, ... g_{0,m_0})$ Hence

$$f_t \in (g_1, ... g_m g_{r-1,1}, ... g_{r-1,m_{r-1}}, ..., g_{0,1}, ..., g_{0,m_0})$$

and so the polynomials $g_1, ..., g_{0,m_0}$ generate a

One of the great successes of category theory in computer science has been the development of a "unified theory" of the constructions underlying denotational semantics. In the untyped λ-calculus, any term may appear in the function position of an application. This means that a model D of the λ-calculus must have the property that given a term t whose interpretation is $d \in D$, Also, the interpretation of a functional abstraction like $\lambda x . x$ is most conveniently defined as a function from $D\ to\ D$, which must then be regarded as an element of D. Let $\psi : [D \to D] \to D$ be the function that picks out elements of D to represent elements of $[D \to D]$ and $\phi : D \to [D \to D]$ be the function that maps elements of *D* to functions of *D*. Since $\psi(f)$ is intended to represent the function f as an element of *D*, it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi o \psi = id_{[D \to D]}$ Furthermore, we often want to view every element of *D* as representing some function from *D to D* and require that elements representing the same function be equal – that is

$$\psi(\varphi(d)) = d$$

or

$$\psi o \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that $\phi \, and \, \psi$ are inverses--- that is, *D* is isomorphic to the space of functions from *D to D* that can be the interpretations of functional abstractions: $D \cong [D \to D]$ .Let us suppose we are working with the untyped $\lambda - calculus$, we need a solution ot the equation $D \cong A + [D \to D]$, where A is some predetermined domain containing interpretations for elements of *C*. Each element of *D* corresponds to either an element of *A* or an element of $[D \to D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \to X]$ from domains to domains --- that is, finding domains *X* such that $X \cong A + [X \to X]$, and such that for any domain *Y* also satisfying this equation, there is an embedding of *X* to *Y* --- a pair of maps

$$X \underset{f^R}{\overset{f}{\rightleftarrows}} Y$$

Such that

$$f^R o f = id_X$$

$$f o f^R \subseteq id_Y$$

Where $f \subseteq g$ means that $f \; approximates \; g$ in some ordering representing their information content. The key shift of perspective from the domain-theoretic to the more general category-theoretic approach lies in considering *F* not as a function on domains, but as a *functor* on a category of domains. Instead of a least fixed point of the function, *F.*

Definition 1.3: Let ***K*** be a category and $F : K \to K$ as a functor. A fixed point of *F* is a pair (A,a), where A is a ***K-object*** and $a : F(A) \to A$ is an isomorphism. A prefixed point of F is a pair (A,a), where A is a ***K-object*** and a is any arrow from F(A) to A

Definition 1.4 : An $\omega - chain$ in a category ***K*** is a diagram of the following form:

$$\Delta = D_o \xrightarrow{f_o} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} .....$$

Recall that a cocone μ of an $\omega - chain \; \Delta$ is a *K*-object *X* and a collection of K *–arrows* $\{\mu_i : D_i \to X \mid i \ge 0\}$ such that $\mu_i = \mu_{i+1} o f_i$ for all $i \ge 0$. We sometimes write $\mu : \Delta \to X$ as a reminder of the arrangement of μ's components Similarly, a colimit $\mu : \Delta \to X$ is a cocone with the property that if $\nu : \Delta \to X'$ is also a cocone then there exists a unique mediating arrow $k : X \to X'$ such that for all $i \ge 0$,, $\nu_i = k \, o \, \mu_i$. Colimits of $\omega - chains$ are sometimes referred to as $\omega - co\lim its$. Dually, an $\omega^{op} - chain$ in ***K*** is a diagram of the following form:

$$\Delta = D_o \xleftarrow{f_o} D_1 \xleftarrow{f_1} D_2 \xleftarrow{f_2} .....$$

A cone $\mu : X \to \Delta$ of an $\omega^{op} - chain \; \Delta$ is a **K**-object X and a collection of **K**-arrows $\{\mu_i : D_i \mid i \ge 0\}$ such that for all $i \ge 0$, $\mu_i = f_i \, o \, \mu_{i+1}$. An ω^{op} -limit of an $\omega^{op} - chain \; \Delta$ is a cone $\mu : X \to \Delta$ with the property that if $\nu : X' \to \Delta$ is also a cone, then there exists a unique mediating arrow $k : X' \to X$ such that for all $i \ge 0, \mu_i \, o k = \nu_i$. We write $\perp_k$ (or just $\perp$) for the distinguish initial object of ***K,*** when it has one, and $\perp \to A$ for the unique arrow from $\perp$ to each ***K***-object A. It is also convenient to write $\Delta^- = D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} .....$ to denote all of Δ except D_o and f_0. By analogy, μ^- is $\{\mu_i \mid i \ge 1\}$. For the images of Δ and μ under ***F*** we write

$$F(\Delta) = F(D_o) \xrightarrow{F(f_o)} F(D_1) \xrightarrow{F(f_1)} F(D_2) \xrightarrow{F(f_2)} .....$$

and $F(\mu) = \{F(\mu_i) \mid i \ge 0\}$

We write F^i for the **i**-fold iterated composition of ***F*** – that is, $F^o(f) = f, F^1(f) = F(f), F^2(f) = F(F(f))$,etc. With these definitions we can state that every monitonic function on a complete lattice has a least fixed point:

Lemma 1.4. Let K be a category with initial object $\perp$ and let $F: K \to K$ be a functor. Define the $\omega - chain \Delta$ by

$$\Delta = \perp \xrightarrow{!\perp \to F(\perp)} F(\perp) \xrightarrow{F(!\perp \to F(\perp))} F^2(\perp) \xrightarrow{F^2(!\perp \to F(\perp))} \ldots\ldots\ldots$$

If both $\mu : \Delta \to D$ and $F(\mu): F(\Delta) \to F(D)$ are colimits, then (D,d) is an intial F-algebra, where $d: F(D) \to D$ is the mediating arrow from $F(\mu)$ to the cocone μ^-

Theorem 1.4 Let a DAG G given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in G be specified. Then the product of these conditional distributions yields a joint probability distribution P of the variables, and (G,P) satisfies the Markov condition.

Proof. Order the nodes according to an ancestral ordering. Let $X_1, X_2, \ldots\ldots X_n$ be the resultant ordering. Next define.

$$P(x_1, x_2, \ldots. x_n) = P(x_n \mid pa_n) P(x_{n-1} \mid Pa_{n-1}) \ldots$$
$$..P(x_2 \mid pa_2) P(x_1 \mid pa_1),$$

Where PA_i is the set of parents of X_i of in G and $P(x_i \mid pa_i)$ is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, $0 \le P(x_1, x_2, \ldots x_n) \le 1$ for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the Markov condition is satisfied. To do this, we need show for $1 \le k \le n$ that
whenever

$$P(pa_k) \ne 0, if \;\; P(nd_k \mid pa_k) \ne 0$$
$$and \;\; P(x_k \mid pa_k) \ne 0$$
$$then \; P(x_k \mid nd_k, pa_k) = P(x_k \mid pa_k),$$

Where ND_k is the set of nondescendents of X_k of in G. Since $PA_k \subseteq ND_k$, we need only show $P(x_k \mid nd_k) = P(x_k \mid pa_k)$. First for a given k, order the nodes so that all and only nondescendents of X_k precede X_k in the ordering. Note that this ordering depends on k, whereas the ordering in the first part of the proof does not. Clearly then

$$ND_k = \{X_1, X_2, \ldots. X_{k-1}\}$$
$$Let$$
$$D_k = \{X_{k+1}, X_{k+2}, \ldots. X_n\}$$

follows $\sum_{d_k}$

We define the m^{th} *cyclotomic field to be the field* $Q[x]/(\Phi_m(x))$ Where $\Phi_m(x)$ is the m^{th} cyclotomic polynomial. $Q[x]/(\Phi_m(x))$ $\Phi_m(x)$ *has degree* $\varphi(m)$ *over* Q *since* $\Phi_m(x)$ has degree $\varphi(m)$. *The roots of* $\Phi_m(x)$ are just the primitive m^{th} roots of unity, so the complex embeddings of $Q[x]/(\Phi_m(x))$ *are simply the* $\varphi(m)$ *maps*

$$\sigma_k : Q[x]/(\Phi_m(x)) \mapsto C,$$
$$1 \le k \prec m, (k,m) = 1, \quad where$$
$$\sigma_k(x) = \xi_m^k,$$

ξ_m being our fixed choice of primitive m^{th} root of unity. Note that $\xi_m^k \in Q(\xi_m)$ for every k; it follows that $Q(\xi_m) = Q(\xi_m^k)$ for all k relatively prime to m. In particular, the images of the σ_i coincide, so $Q[x]/(\Phi_m(x))$ *is Galois over* Q. *This means that we can write* $Q(\xi_m)$ *for* $Q[x]/(\Phi_m(x))$ *without much fear of ambiguity; we will do so from now on, the identification being* $\xi_m \mapsto x$. *One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another,or intersections or compositums; all of these things take place considering them as subfield of* C. We now investigate some basic properties of cyclotomic fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in $Q(\xi_m)$ .Note, for example, that if m is odd, then $-\xi_m$ is a $2m^{th}$ root of unity. We will show that this is the only way in which one can obtain any non-m^{th} roots of unity.

LEMMA 1.5 If m divides n , then $Q(\xi_m)$ *is contained in* $Q(\xi_n)$

PROOF. Since $\xi^{n/m} = \xi_m$, *we have* $\xi_m \in Q(\xi_n)$, *so the result is clear*

LEMMA 1.6 If m and n are relatively prime, then

$$Q(\xi_m,\xi_n)=Q(\xi_{nm})$$

and

$$Q(\xi_m)\cap Q(\xi_n)=Q$$

(Recall the $Q(\xi_m,\xi_n)$ is the compositum of $Q(\xi_m)$ *and* $Q(\xi_n)$)

PROOF. One checks easily that $\xi_m\xi_n$ is a primitive mn^{th} root of unity, so that

$$Q(\xi_{mn})\subseteq Q(\xi_m,\xi_n)$$

$$\left[Q(\xi_m,\xi_n):Q\right]\le\left[Q(\xi_m):Q\right]\left[Q(\xi_n:Q\right]$$

$$=\varphi(m)\varphi(n)=\varphi(mn);$$

Since $\left[Q(\xi_{mn}):Q\right]=\varphi(mn);$ this implies that $Q(\xi_m,\xi_n)=Q(\xi_{nm})$ We know that $Q(\xi_m,\xi_n)$ has degree $\varphi(mn)$ over Q, so we must have

$$\left[Q(\xi_m,\xi_n):Q(\xi_m)\right]=\varphi(n)$$

and

$$\left[Q(\xi_m,\xi_n):Q(\xi_m)\right]=\varphi(m)$$

$$\left[Q(\xi_m):Q(\xi_m)\cap Q(\xi_n)\right]\ge\varphi(m)$$

And thus that $Q(\xi_m)\cap Q(\xi_n)=Q$

PROPOSITION 1.2 For any m and n

$$Q(\xi_m,\xi_n)=Q(\xi_{[m,n]})$$

And

$$Q(\xi_m)\cap Q(\xi_n)=Q(\xi_{(m,n)});$$

here $[m,n]$ and (m,n) denote the least common multiple and the greatest common divisor of m and n, respectively.

PROOF. Write $m=p_1^{e_1}......p_k^{e_k}$ *and* $p_1^{f_1}....p_k^{f_k}$ where the p_i are distinct primes. (We allow e_i *or* f_i to be zero)

$$Q(\xi_m)=Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}})...Q(\xi_{p_k^{e_k}})$$

and

$$Q(\xi_n)=Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}})...Q(\xi_{p_k^{f_k}})$$

Thus

$$\begin{aligned}Q(\xi_m,\xi_n)&=Q(\xi_{p_1^{e_1}})........Q(\xi_{p_2^{e_k}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{f_k}})\\&=Q(\xi_{p_1^{e_1}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{e_k}})Q(\xi_{p_k^{f_k}})\\&=Q(\xi_{p_1^{\max(e_1,f_1)}}).......Q(\xi_{p_1^{\max(e_k,f_k)}})\\&=Q(\xi_{p_1^{\max(e_1,f_1)}........p_1^{\max(e_k,f_k)}})\\&=Q(\xi_{[m,n]});\end{aligned}$$

An entirely similar computation shows that $Q(\xi_m)\cap Q(\xi_n)=Q(\xi_{(m,n)})$

Mutual information measures the information transferred when x_i is sent and y_i is received, and is defined as

$$I(x_i,y_i)=\log_2\frac{P(x_i/y_i)}{P(x_i)}\ bits \qquad (1)$$

In a noise-free channel, **each** y_i is uniquely connected to the corresponding x_i, and so they constitute an input –output pair (x_i,y_i) for which

$$P(x_i/y_j)=1 \text{ and } I(x_i,y_j)=\log_2\frac{1}{P(x_i)} \text{ bits;}$$

that is, the transferred information is equal to the self-information that corresponds to the input x_i In a very noisy channel, the output y_i and input x_i would be completely uncorrelated, and so $P(x_i/y_j)=P(x_i)$ and also $I(x_i,y_j)=0;$ that is, there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X,Y)=\sum_{i.j}P(x_i,y_j)I(x_i,y_j)=\sum_{i.j}P(x_i,y_j)\log_2\left[\frac{P(x_i/y_j)}{P(x_i)}\right]$$

bits per symbol . This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i, y_j) = P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

$$P(y_j) = \sum_i P(y_j/x_i)P(x_i)$$

$$P(x_i) = \sum_i P(x_i/y_j)P(y_j)$$

Then

$$I(X,Y) = \sum_{i.j} P(x_i, y_j)$$

$$= \sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i)}\right]$$

$$-\sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i/y_j)}\right]$$

$$\sum_{i.j} P(x_i, y_j)\log_2\left[\frac{1}{P(x_i)}\right]$$

$$= \sum_i\left[P(x_i/y_j)P(y_j)\right]\log_2\frac{1}{P(x_i)}$$

$$\sum_i P(x_i)\log_2\frac{1}{P(x_i)} = H(X)$$

$$I(X,Y) = H(X) - H(X/Y)$$

Where $H(X/Y) = \sum_{i,j} P(x_i, y_j)\log_2\frac{1}{P(x_i/y_j)}$

is usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol y_j provides $H(X) - H(X/Y)$ bits of information. This difference is the mutual information of the channel. *Mutual Information: Properties* Since

$$P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

The mutual information fits the condition

$$I(X,Y) = I(Y,X)$$

And by interchanging input and output it is also true that

$$I(X,Y) = H(Y) - H(Y/X)$$

Where

$$H(Y) = \sum_j P(y_j)\log_2\frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol after knowing the corresponding output symbol

$$I(X,Y) = H(X) - H(X/Y)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and is spite of the fact that for some $y_j, H(X/y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i, y_j)\log_2\frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i, y_j)\log_2\frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

Then

$$-I(X,Y) = \sum_{i,j} P(x_i, y_j)\frac{P(x_i)P(y_j)}{P(x_i, y_j)} \le 0$$

Because this expression is of the form

$$\sum_{i=1}^{M} P_i\log_2\left(\frac{Q_i}{P_i}\right) \le 0$$

The above expression can be applied due to the factor $P(x_i)P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity Q_i, which in this expression is a dummy variable that fits the condition $\sum_i Q_i \le 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$H(X,Y) = \sum_{i,j} P(x_i, y_j)\log_2\frac{1}{P(x_i, y_j)}$$

$$= \sum_{i,j} P(x_i, y_j)\log_2\frac{P(x_i)P(y_j)}{P(x_i, y_j)}$$

$$+\sum_{i,j} P(x_i, y_j)\log_2\frac{1}{P(x_i)P(y_j)}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$P(x_1) = \alpha$ *and* $P(x_2) = 1 - \alpha$, and transition probabilities

$P(y_3/x_2) = 1-p$ and $P(y_2/x_1) = 0,$

and $P(y_3/x_1) = 0$

and $P(y_1/x_2) = p$

and $P(y_3/x_2) = 1-p$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s, let n be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean *n*-space. $p(y|x)$ for the density $p_n(y_1,...,y_n | x_1,...x_n)$ and F for F_n. For any real number a, let

$$A = \left\{ (x,y) : \log \frac{p(y|x)}{p(y)} > a \right\} \qquad (1)$$

Then for each positive integer u, there is a code (u,n,λ) such that

$$\lambda \le ue^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\} \qquad (2)$$

Where

$P\{(X,Y)\in A\} = \int_A ... \int p(x,y)dxdy, \qquad p(x,y) = p(x)p(y|x)$

and

$$P\{X \in F\} = \int_F ... \int p(x)dx$$

Proof: A sequence $x^{(1)} \in F$ *such that*

$$P\{Y \in A_{x^1} | X = x^{(1)}\} \ge 1-\varepsilon$$

$$where\ A_x = \{y : (x,y) \varepsilon A\};$$

Choose the decoding set B_1 to be $A_{x^{(1)}}$. Having chosen $x^{(1)},........,x^{(k-1)}$ and $B_1,...,B_{k-1}$, select $x^k \in F$ such that

$$P\left\{ Y \in A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i | X = x^{(k)} \right\} \ge 1-\varepsilon;$$

Set $B_k = A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i$, If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets $B_i,\ i = 1,2,...,u,$ form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t = 0$). We will show $t \ge u$ by showing that $\varepsilon \le te^{-a} + P\{(X,Y) \notin A\} + P\{X \notin F\}$. We proceed as follows.

Let

$$B = \bigcup_{j=1}^{t} B_j. \quad (If\ \ t=0,\ take\ \ B=\phi).\ \ Then$$

$$P\{(X,Y) \in A\} = \int_{(x,y)\in A} p(x,y)dx\,dy$$

$$= \int_x p(x) \int_{y\in A_x} p(y|x)dy\,dx$$

$$= \int_x p(x) \int_{y\in B\cap A_x} p(y|x)dy\,dx + \int_x p(x)$$

C. Algorithms

Ideals. Let A be a ring. Recall that an *ideal a* in A is a subset such that a is subgroup of A regarded as a group under addition;

$$a \in a, r \in A \Rightarrow ra \in A$$

The ideal generated by a subset S of A is the intersection of all ideals A containing a ----- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form $\sum r_i s_i$ with $r_i \in A, s_i \in S$. When $S = \{s_1,.....,s_m\}$, we shall write $(s_1,.....,s_m)$ for the ideal it generates.

Let a and b be ideals in A. The set $\{a+b | a \in a, b \in b\}$ is an ideal, denoted by $a+b$. The ideal generated by $\{ab | a \in a, b \in b\}$ is denoted by ab. Note that $ab \subset a \cap b$. Clearly ab consists of all finite sums $\sum a_i b_i$ with $a_i \in a$ and $b_i \in b$, and if $a = (a_1,...,a_m)$ and $b = (b_1,...,b_n)$, then $ab = (a_1 b_1,...,a_i b_j,...,a_m b_n)$ .Let a be an ideal of A. The set of cosets of a in A forms a ring A/a, and $a \mapsto a+a$ is a homomorphism $\phi : A \mapsto A/a$. The map $b \mapsto \phi^{-1}(b)$ is a one to one correspondence between the ideals of A/a and the ideals of A containing a An ideal p if *prime* if $p \ne A$ and $ab \in p \Rightarrow a \in p$ or $b \in p$. Thus p is prime if and only if A/p is nonzero and has the property that $ab = 0, \quad b \ne 0 \Rightarrow a = 0,$ i.e., A/p is an integral domain. An ideal m is *maximal* if $m \ne| A$ and there does not exist an ideal n contained strictly between m and A. Thus m is maximal if and only if A/m has no proper nonzero ideals, and so is a field. Note that m maximal $\Rightarrow$ m prime. The ideals of $A \times B$ are all of the form $a \times b$, with a and b ideals in A and B. To see this, note that if c is an ideal in $A \times B$ and

$(a,b)\in c$, then $(a,0)=(a,b)(1,0)\in c$ and $(0,b)=(a,b)(0,1)\in c$. This shows that $c=a\times b$ with

$a=\{a\,|\,(a,b)\in c\ \ some\ \ b\in b\}$

and

$b=\{b\,|\,(a,b)\in c\ \ some\ \ a\in a\}$

Let A be a ring. An A-algebra is a ring B together with a homomorphism $i_B : A\rightarrow B$. A *homomorphism of* A -algebra $B\rightarrow C$ is a homomorphism of rings $\varphi : B\rightarrow C$ such that $\varphi(i_B(a))=i_C(a)$ for all $a\in A$. An A-algebra B is said to be *finitely generated* (or of *finite-type* over A) if there exist elements $x_1,...,x_n\in B$ such that every element of B can be expressed as a polynomial in the x_i with coefficients in $i(A)$, i.e., such that the homomorphism $A[X_1,...,X_n]\rightarrow B$ sending X_i to x_i is surjective. A ring homomorphism $A\rightarrow B$ is *finite*, and B is finitely generated as an A-module. Let k be a field, and let A be a k -algebra. If $1\neq 0$ in A, then the map $k\rightarrow A$ is injective, we can identify k with its image, i.e., we can regard k as a subring of A . If 1=0 in a ring R, the R is the zero ring, i.e., $R=\{0\}$. **Polynomial rings.** Let k be a field. A *monomial* in $X_1,...,X_n$ is an expression of the form $X_1^{a_1}...X_n^{a_n}$, $a_j\in N$. The *total degree* of the monomial is $\sum a_i$. We sometimes abbreviate it by X^{α}, $\alpha=(a_1,...,a_n)\in \mathbb{N}^n$. The elements of the polynomial ring $k[X_1,...,X_n]$ are finite sums $\sum c_{a_1....a_n}X_1^{a_1}...X_n^{a_n}$, $c_{a_1....a_n}\in k$, $a_j\in\mathbb{N}$
With the obvious notions of equality, addition and multiplication. Thus the monomials from basis for $k[X_1,...,X_n]$ as a k -vector space. The ring $k[X_1,...,X_n]$ is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial $f(X_1,...,X_n)$ is *irreducible* if it is nonconstant and has only the obvious factorizations, i.e., $f=gh\Rightarrow g$ or h is constant. **Division in** $k[X]$. The division algorithm allows us to divide a nonzero polynomial into another: let f and g be polynomials in $k[X]$ with $g\neq 0$; then there exist unique polynomials $q,r\in k[X]$ such that $f=qg+r$ with either $r=0$ or $\deg r<\deg g$. Moreover, there is an algorithm for deciding whether $f\in(g)$, namely, find r and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in $k[X]$ to a single generator by successively replacing each pair of generators with their greatest common divisor.

(Pure) ***lexicographic*** *ordering (lex)*. Here monomials are ordered by lexicographic(dictionary) order. More precisely, let $\alpha=(a_1,...a_n)$ and $\beta=(b_1,...b_n)$ be two elements of $\mathbb{N}^n$; then $\alpha>\beta$ *and* $X^{\alpha}>X^{\beta}$ (lexicographic ordering) if, in the vector difference $\alpha-\beta\in\mathbb{Z}$, the left most nonzero entry is positive. For example, $XY^2>Y^3Z^4$; $X^3Y^2Z^4>X^3Y^2Z$. Note that this isn't quite how the dictionary would order them: it would put *XXXYYZZZZ* after *XXXYYZ* . *Graded reverse lexicographic order (grevlex)*. Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, $\alpha>\beta$ if $\sum a_i>\sum b_i$, or $\sum a_i=\sum b_i$ and in $\alpha-\beta$ the right most nonzero entry is negative. For example:

$X^4Y^4Z^7>X^5Y^5Z^4$ *(total degree greater)*

$XY^5Z^2>X^4YZ^3,\quad X^5YZ>X^4YZ^2$.

Orderings on $k[X_1,...X_n]$. Fix an ordering on the monomials in $k[X_1,...X_n]$. Then we can write an element f of $k[X_1,...X_n]$ in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write

$$f=4XY^2Z+4Z^2-5X^3+7X^2Z^2$$

as

$$f=-5X^3+7X^2Z^2+4XY^2Z+4Z^2\quad (lex)$$

or

$$f=4XY^2Z+7X^2Z^2-5X^3+4Z^2\quad (grevlex)$$

Let $\sum a_{\alpha}X^{\alpha}\in k[X_1,...,X_n]$, in decreasing order:

$$f=a_{\alpha_0}X^{\alpha_0}+_{\alpha_1}X^{\alpha_1}+...,\qquad \alpha_0>\alpha_1>...,\quad \alpha_0\neq 0$$

Then we define.

- The *multidegree* of f to be multdeg(f)= α_0 ;

- The *leading coefficient of* f to be *LC(* f *)=* a_{α_0} *;*
- The *leading monomial of* f to be *LM(* f *)* = X^{α_0} *;*
- The *leading term of* f to be *LT(* f *)* = $a_{\alpha_0} X^{\alpha_0}$

For the polynomial $f = 4XY^2Z + ...,$ the multidegree is (1,2,1), the leading coefficient is 4, the leading monomial is XY^2Z, and the leading term is $4XY^2Z$. **The division algorithm in** $k[X_1,...X_n]$**.** Fix a monomial ordering in $\mathbb{Z}^2$. Suppose given a polynomial f and an ordered set $(g_1,...g_s)$ of polynomials; the division algorithm then constructs polynomials $a_1,...a_s$ and r such that $f = a_1g_1 + ... + a_sg_s + r$ Where either $r = 0$ or no monomial in r is divisible by any of $LT(g_1),...,LT(g_s)$ **Step 1:** If $LT(g_1)|LT(f)$, divide g_1 into f to get

$$f = a_1g_1 + h, \qquad a_1 = \frac{LT(f)}{LT(g_1)} \in k[X_1,...,X_n]$$

If $LT(g_1)|LT(h)$, repeat the process until $f = a_1g_1 + f_1$ (different a_1) with $LT(f_1)$ not divisible by $LT(g_1)$. Now divide g_2 into f_1, and so on, until $f = a_1g_1 + ... + a_sg_s + r_1$ With $LT(r_1)$ not divisible by any $LT(g_1),...LT(g_s)$ **Step 2:** Rewrite $r_1 = LT(r_1) + r_2$, and repeat Step 1 with r_2 for f : $f = a_1g_1 + ... + a_sg_s + LT(r_1) + r_3$ (different a_i's) **Monomial ideals.** In general, an ideal a will contain a polynomial without containing the individual terms of the polynomial; for example, the ideal $a = (Y^2 - X^3)$ contains $Y^2 - X^3$ but not Y^2 or X^3.

DEFINITION 1.5. An ideal a is *monomial* if $\sum c_\alpha X^\alpha \in a \Rightarrow X^\alpha \in a$ all α with $c_\alpha \neq 0$.

PROPOSITION 1.3. Let a be a *monomial ideal*, and let $A = \{\alpha \mid X^\alpha \in a\}$. Then A satisfies the condition $\alpha \in A, \ \beta \in \mathbb{Z}^n \Rightarrow \alpha + \beta \in$ $(*)$ And a is the k-subspace of $k[X_1,...,X_n]$ generated by the $X^\alpha, \alpha \in A$. Conversely, of A is a subset of $\mathbb{Z}^n$ satisfying $(*)$, then the k-subspace a of $k[X_1,...,X_n]$ generated by $\{X^\alpha \mid \alpha \in A\}$ is a monomial ideal.

PROOF. It is clear from its definition that a monomial ideal a is the k-subspace of $k[X_1,...,X_n]$ generated by the set of monomials it contains. If $X^\alpha \in a$ and $X^\beta \in k[X_1,...,X_n]$.

If a permutation is chosen uniformly and at random from the $n!$ possible permutations in S_n, then the counts $C_j^{(n)}$ of cycles of length j are dependent random variables. The joint distribution of $C^{(n)} = (C_1^{(n)},...,C_n^{(n)})$ follows from Cauchy's formula, and is given by

$$P[C^{(n)} = c] = \frac{1}{n!}N(n,c) = 1\left\{\sum_{j=1}^{n} jc_j = n\right\}\prod_{j=1}^{n}\left(\frac{1}{j}\right)^{c_j}\frac{1}{c_j!}, \qquad (1.1)$$

for $c \in \mathbb{Z}_+^n$.

Lemma1.7 For nonnegative integers $m_{1,...,}m_n$,

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right) = \left(\prod_{j=1}^{n}\left(\frac{1}{j}\right)^{m_j}\right)1\left\{\sum_{j=1}^{n} jm_j \le n\right\} \qquad (1.4)$$

Proof. This can be established directly by exploiting cancellation of the form $c_j^{[m_j]}/c_j^! = 1/(c_j - m_j)!$ when $c_j \ge m_j$, which occurs between the ingredients in Cauchy's formula and the falling factorials in the moments. Write $m = \sum jm_j$. Then, with the first sum indexed by $c = (c_1,...c_n) \in \mathbb{Z}_+^n$ and the last sum indexed by $d = (d_1,...,d_n) \in \mathbb{Z}_+^n$ via the correspondence $d_j = c_j - m_j$, we have

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right) = \sum_c P[C^{(n)} = c]\prod_{j=1}^{n}(c_j)^{[m_j]}$$

$$= \sum_{c: c_j \ge m_j \text{ for all } j} 1\left\{\sum_{j=1}^{n} jc_j = n\right\}\prod_{j=1}^{n}\frac{(c_j)^{[m_j]}}{j^{c_j}c_j!}$$

$$= \prod_{j=1}^{n}\frac{1}{j^{m_j}}\sum_d 1\left\{\sum_{j=1}^{n} jd_j = n - m\right\}\prod_{j=1}^{n}\frac{1}{j^{d_j}(d_j)!}$$

This last sum simplifies to the indicator $1(m \le n)$, corresponding to the fact that if $n - m \ge 0$, then

$d_j = 0$ for $j > n-m$, and a random permutation in S_{n-m} must have some cycle structure $(d_1,...,d_{n-m})$. The moments of $C_j^{(n)}$ follow immediately as

$$E(C_j^{(n)})^{[r]} = j^{-r} 1\{jr \le n\} \quad (1.2)$$

We note for future reference that (1.4) can also be written in the form

$$E\left(\prod_{j=1}^{n} (C_j^{(n)})^{[m_j]}\right) = E\left(\prod_{j=1}^{n} Z_j^{[m_j]}\right) 1\left\{\sum_{j=1}^{n} jm_j \le n\right\}, \quad (1.3)$$

Where the Z_j are independent Poisson-distribution random variables that satisfy $E(Z_j) = 1/j$

The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts C_j^n, we find the distribution of C_j^n using a combinatorial approach combined with the inclusion-exclusion formula.

Lemma 1.8. For $1 \le j \le n$,

$$P[C_j^{(n)} = k] = \frac{j^{-k}}{k!} \sum_{l=0}^{[n/j]-k} (-1)^l \frac{j^{-l}}{l!} \quad (1.1)$$

Proof. Consider the set I of all possible cycles of length j, formed with elements chosen from $\{1, 2, ...n\}$, so that $|I| = n^{[j]/j}$. For each $\alpha \in I$, consider the "property" G_α of having α; that is, G_α is the set of permutations $\pi \in S_n$ such that α is one of the cycles of π. We then have $|G_\alpha| = (n-j)!$, since the elements of $\{1, 2, ..., n\}$ not in α must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term S_r, which is the sum of the probabilities of the r-fold intersection of properties, summing over all sets of r distinct properties. There are two cases to consider. If the r properties are indexed by r cycles having no elements in common, then the intersection specifies how rj elements are moved by the permutation, and there are $(n-rj)!1(rj \le n)$ permutations in the intersection. There are $n^{[rj]}/(j^r r!)$ such intersections. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the r-fold intersection is empty. Thus

$$S_r = (n-rj)!1(rj \le n) \times \frac{n^{[rj]}}{j^r r!} \frac{1}{n!} = 1(rj \le n) \frac{1}{j^r r!}$$

Finally, the inclusion-exclusion series for the number of permutations having exactly k properties is

$$\sum_{l \ge 0} (-1)^l \binom{k+l}{l} S_{k+l},$$

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute j=1 in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For $k = 0, 1, ..., n,$

$$P[C_1^{(n)} = k] = \frac{1}{k!} \sum_{l=0}^{n-k} (-1)^l \frac{1}{l!}, \quad (1.2)$$

and the moments of $C_1^{(n)}$ follow from (1.2) with $j = 1$. In particular, for $n \ge 2$, the mean and variance of $C_1^{(n)}$ are both equal to 1. The joint distribution of $(C_1^{(n)}, ..., C_b^{(n)})$ for any $1 \le b \le n$ has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any $c = (c_1, ..., c_b) \in \mathbb{N}_+^b$ with $m = \sum ic_i$,

$$P[(C_1^{(n)}, ..., C_b^{(n)}) = c]$$
$$= \left\{\prod_{i=1}^{b} \left(\frac{1}{i}\right)^{c_i} \frac{1}{c_i!}\right\} \sum_{\substack{l \ge 0 \text{ with} \\ \sum il_i \le n-m}} (-1)^{l_1+...+l_b} \prod_{i=1}^{b} \left(\frac{1}{i}\right)^{l_i} \frac{1}{l_i!} \quad (1.3)$$

The joint moments of the first b counts $C_1^{(n)}, ..., C_b^{(n)}$ can be obtained directly from (1.2) and (1.3) by setting $m_{b+1} = ... = m_n = 0$

The limit distribution of cycle counts

It follows immediately from Lemma 1.2 that for each fixed j, as $n \to \infty$,

$$P[C_j^{(n)} = k] \to \frac{j^{-k}}{k!} e^{-1/j}, \quad k = 0, 1, 2, ...,$$

So that $C_j^{(n)}$ converges in distribution to a random variable Z_j having a Poisson distribution with mean $1/j$; we use the notation $C_j^{(n)} \to_d Z_j$ where $Z_j \sim P_o(1/j)$ to describe this. Infact, the limit random variables are independent.

Theorem 1.6 The process of cycle counts converges in distribution to a Poisson process of $\mathbb{N}$ with intensity j^{-1}. That is, as $n \to \infty$,

$$(C_1^{(n)}, C_2^{(n)}, ...) \to_d (Z_1, Z_2, ...) \quad (1.1)$$

Where the $Z_j, j=1,2,...,$ are independent Poisson-distributed random variables with $E(Z_j)=\frac{1}{j}$

Proof. To establish the converges in distribution one shows that for each fixed $b\ge 1,$ as $n\to\infty,$

$$P[(C_1^{(n)},...,C_b^{(n)})=c]\to P[(Z_1,...,Z_b)=c]$$

Error rates

The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when $b=1$. Using properties of alternating series with decreasing terms, for $k=0,1,...,n,$

$$\frac{1}{k!}\left(\frac{1}{(n-k+1)!}-\frac{1}{(n-k+2)!}\right)\le\left|P[C_1^{(n)}=k]-P[Z_1=k]\right|$$
$$\le\frac{1}{k!(n-k+1)!}$$

It follows that

$$\frac{2^{n+1}}{(n+1)!}\frac{n}{n+2}\le\sum_{k=0}^{n}\left|P[C_1^{(n)}=k]-P[Z_1=k]\right|\le\frac{2^{n+1}-1}{(n+1)!}\quad(1.11)$$

Since

$$P[Z_1>n]=\frac{e^{-1}}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+...\right)<\frac{1}{(n+1)!},$$

We see from (1.11) that the total variation distance between the distribution $L(C_1^{(n)})$ of $C_1^{(n)}$ and the distribution $L(Z_1)$ of Z_1

Establish the asymptotics of $P\left[A_n(C^{(n)})\right]$ under conditions (A_0) and (B_{01}), where

$$A_n(C^{(n)})=\bigcap_{1\le i\le n}\ \bigcap_{r_i^{'}+1\le j\le r_i}\left\{C_{ij}^{(n)}=0\right\},$$

and $\zeta_i=(r_i^{'}/r_{id})-1=O(i^{-g^{'}})$ as $i\to\infty,$ for some $g^{'}>0.$ We start with the expression

$$P[A_n(C^{(n)})]=\frac{P[T_{0m}(Z^{'})=n]}{P[T_{0m}(Z)=n]}$$

$$\prod_{\substack{1\le i\le n\\ r_i^{'}+1\le j\le r_i}}\left\{1-\frac{\theta}{ir_i}(1+E_{i0})\right\}\quad(1.1)$$

$$P[T_{0n}(Z^{'})=n]$$
$$=\frac{\theta d}{n}\exp\left\{\sum_{i\ge 1}[\log(1+i^{-1}\theta d)-i^{-1}\theta d]\right\}$$
$$\left\{1+O(n^{-1}\varphi_{\{1,2,7\}}^{'}(n))\right\}\quad(1.2)$$

and

$$P[T_{0n}(Z^{'})=n]$$
$$=\frac{\theta d}{n}\exp\left\{\sum_{i\ge 1}[\log(1+i^{-1}\theta d)-i^{-1}\theta d]\right\}$$
$$\left\{1+O(n^{-1}\varphi_{\{1,2,7\}}(n))\right\}\quad(1.3)$$

Where $\varphi_{\{1,2,7\}}^{'}(n)$ refers to the quantity derived from $Z^{'}$. It thus follows that $P[A_n(C^{(n)})]\sim Kn^{-\theta(1-d)}$ for a constant K, depending on Z and the $r_i^{'}$ and computable explicitly from (1.1) – (1.3), if Conditions (A_0) and (B_{01}) are satisfied and if $\zeta_i^*=O(i^{-g^{'}})$ from some $g^{'}>0,$ since, under these circumstances, both $n^{-1}\varphi_{\{1,2,7\}}^{'}(n)$ and $n^{-1}\varphi_{\{1,2,7\}}(n)$ tend to zero as $n\to\infty.$ In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order n^{-1} if $g^{'}>1.$

For $0\le b\le n/8$ and $n\ge n_0,$ with n_0

$$d_{TV}(L(C[1,b]),L(Z[1,b]))$$
$$\le d_{TV}(L(\tilde{C}[1,b]),L(\tilde{Z}[1,b]))$$
$$\le\varepsilon_{\{7,7\}}(n,b),$$

Where $\varepsilon_{\{7,7\}}(n,b)=O(b/n)$ under Conditions $(A_0),(D_1)$ and (B_{11}) Since, by the Conditioning Relation,

$$L(\tilde{C}[1,b]\,|\,T_{0b}(C)=l)=L(\tilde{Z}[1,b]\,|\,T_{0b}(Z)=l),$$

It follows by direct calculation that

$$d_{TV}(L(\tilde{C}[1,b]),L(\tilde{Z}[1,b]))$$
$$=d_{TV}(L(T_{0b}(C)),L(T_{0b}(Z)))$$
$$=\max_A\sum_{r\in A}P[T_{0b}(Z)=r]$$
$$\left\{1-\frac{P[T_{bn}(Z)=n-r]}{P[T_{0n}(Z)=n]}\right\}\quad(1.4)$$

Suppressing the argument Z from now on, we thus obtain

$$d_{TV}(L(\breve{C}[1,b]), L(\breve{Z}[1,b]))$$
$$= \sum_{r\ge 0} P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+$$
$$\le \sum_{r>n/2} P[T_{0b}=r] + \sum_{r=0}^{[n/2]} \frac{P[T_{0b}=r]}{P[T_{0b}=n]}$$
$$\times\left\{\sum_{s=0}^{n} P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r]\right\}_+$$
$$\le \sum_{r>n/2} P[T_{0b}=r] + \sum_{r=0}^{[n/2]} P[T_{0b}=r]$$
$$\times \sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{\{P[T_{bn}=n-s]-P[T_{bn}=n-r]\}}{P[T_{0n}=n]}$$
$$+\sum_{s=0}^{[n/2]} P[T_{0b}=r]\sum_{s=[n/2]+1}^{n} P[T=s]P[T_{bn}=n-s]/P[T_{0n}=n]$$

The first sum is at most $2n^{-1}ET_{0b}$; the third is bound by

$$(\max_{n/2<s\le n} P[T_{0b}=s])/P[T_{0n}=n]$$
$$\le \frac{2\varepsilon_{\{10.5(1)\}}(n/2,b)}{n}\frac{3n}{\theta P_\theta[0,1]},$$
$$\frac{3n}{\theta P_\theta[0,1]} 4n^{-2}\phi^*_{\{10.8\}}(n)\sum_{r=0}^{[n/2]} P[T_{0b}=r]\sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{1}{2}|r-s|$$
$$\le \frac{12\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\frac{ET_{0b}}{n}$$

Hence we may take

$$\varepsilon_{\{7,7\}}(n,b) = 2n^{-1}ET_{0b}(Z)\left\{1+\frac{6\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\right\}P$$
$$+\frac{6}{\theta P_\theta[0,1]}\varepsilon_{\{10.5(1)\}}(n/2,b) \qquad (1.5)$$

Required order under Conditions $(A_0),(D_1)$ and (B_{11}), if $S(\infty)<\infty$. If not, $\phi^*_{\{10.8\}}(n)$ can be replaced by $\phi^*_{\{10.11\}}(n)$ in the above, which has the required order, without the restriction on the r_i implied by $S(\infty)<\infty$. Examining the Conditions $(A_0),(D_1)$ and (B_{11}), it is perhaps surprising to find that (B_{11}) is required instead of just (B_{01}); that is, that we should need $\sum_{l\ge 2} l\varepsilon_{il} = O(i^{-a_1})$ to hold for some $a_1>1$. A first observation is that a similar problem arises with the rate of decay of ε_{i1} as well. For this reason, n_1 is replaced by $\breve{n}_1$. This makes it possible to replace condition (A_1) by the weaker pair of conditions (A_0) and (D_1) in the eventual assumptions needed for $\varepsilon_{\{7,7\}}(n,b)$ to be of order $O(b/n)$; the decay rate requirement of order $i^{-1-\gamma}$ is shifted from ε_{i1} itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the $\varepsilon_{i1}, l\ge 2$, than are made in (B_{11}). The critical point of the proof is seen where the initial estimate of the difference $P[T_{bn}^{(m)}=s]-P[T_{bn}^{(m)}=s+1]$. The factor $\varepsilon_{\{10.10\}}(n)$, which should be small, contains a far tail element from $\breve{n}_1$ of the form $\phi_1^\theta(n)+u_1^*(n)$, which is only small if $a_1>1$, being otherwise of order $O(n^{1-a_1+\delta})$ for any $\delta>0$, since $a_2>1$ is in any case assumed. For $s\ge n/2$, this gives rise to a contribution of order $O(n^{-1-a_1+\delta})$ in the estimate of the difference $P[T_{bn}=s]-P[T_{bn}=s+1]$, which, in the remainder of the proof, is translated into a contribution of order $O(tn^{-1-a_1+\delta})$ for differences of the form $P[T_{bn}=s]-P[T_{bn}=s+1]$, finally leading to a contribution of order $bn^{-a_1+\delta}$ for any $\delta>0$ in $\varepsilon_{\{7.7\}}(n,b)$. Some improvement would seem to be possible, defining the function g by $g(w)=1_{\{w=s\}}-1_{\{w=s+t\}}$, differences that are of the form $P[T_{bn}=s]-P[T_{bn}=s+t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi_1^\theta(n)+u_1^*(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form

$$|P[T_{bn}=s]-P[T_{bn}=s+t]| = O(n^{-2}t+n^{-1-a_1+\delta})$$

for any $\delta>0$ could perhaps be attained, leading to a final error estimate in order $O(bn^{-1}+n^{-a_1+\delta})$ for any $\delta>0$, to replace $\varepsilon_{\{7.7\}}(n,b)$. This would be of the ideal order $O(b/n)$ for large enough b, but would still be coarser for small b.

With b and n as in the previous section, we wish to show that

$$\left|d_{TV}(L(C[1,b]),L(Z[1,b]))-\frac{1}{2}(n+1)^{-1}|1-\theta|E|T_{0b}-ET_{0b}|\right| \le \varepsilon_{\{7,8\}}(n,b),$$

Where $\varepsilon_{\{7.8\}}(n,b)=O(n^{-1}b[n^{-1}b+n^{-\beta_{12}+\delta}])$ for any $\delta>0$ under Conditions $(A_0),(D_1)$ and (B_{12}), with β_{12}. The proof uses sharper estimates. As before, we begin with the formula

$$d_{TV}(L(C[1,b]),L(Z[1,b])) = \sum_{r\ge 0}P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+$$

Now we observe that

$$\left|\sum_{r\ge 0}P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+ - \sum_{r=0}^{[n/2]}\frac{P[T_{0b}=r]}{P[T_{0n}=n]}\right| \times\left|\sum_{s=[n/2]+1}^{n}P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r])\right|$$

$$\le 4n^{-2}ET_{0b}^2+(\max_{n/2<s\le n}P[T_{0b}=s])/P[T_{0n}=n] +P[T_{0b}>n/2]$$

$$\le 8n^{-2}ET_{0b}^2+\frac{3\varepsilon_{\{10.5(2)\}}(n/2,b)}{\theta P_\theta[0,1]}, \qquad (1.1)$$

We have

$$\left|\sum_{r=0}^{[n/2]}\frac{P[T_{0b}=r]}{P[T_{0n}=n]}\right.$$

$$\times\left(\left\{\sum_{s=0}^{[n/2]}P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r]\right\}_+ - \left\{\sum_{s=0}^{[n/2]}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}P[T_{0n}=n]\right\}_+\right)\Bigg|$$

$$\le\frac{1}{n^2P[T_{0n}=n]}\sum_{r\ge 0}P[T_{0b}=r]\sum_{s\ge 0}P[T_{0b}=s]|s-r| \times\left\{\varepsilon_{\{10.14\}}(n,b)+2(r\vee s)|1-\theta|n^{-1}\left\{K_0\theta+4\phi^*_{\{10.8\}}(n)\right\}\right\}$$

$$\le\frac{6}{\theta nP_\theta[0,1]}ET_{0b}\varepsilon_{\{10.14\}}(n,b) +4|1-\theta|n^{-2}ET_{0b}^2\left\{K_0\theta+4\phi^*_{\{10.8\}}(n)\right\} \left(\frac{3}{\theta nP_\theta[0,1]}\right)\Big\}, \qquad (1.2)$$

The approximation in (1.2) is further simplified by noting that

$$\sum_{r=0}^{[n/2]}P[T_{0b}=r]\left|\left\{\sum_{s=0}^{[n/2]}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}_+ - \left\{\sum_{s=0}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}_+\right|$$

$$\le\sum_{r=0}^{[n/2]}P[T_{0b}=r]\sum_{s>[n/2]}P[T_{0b}=s]\frac{(s-r)|1-\theta|}{n+1}$$

$$\le|1-\theta|n^{-1}E(T_{0b}1\{T_{0b}>n/2\})\le 2|1-\theta|n^{-2}ET_{0b}^2, \qquad (1.3)$$

and then by observing that

$$\sum_{r>[n/2]}P[T_{0b}=r]\left\{\sum_{s\ge 0}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}$$

$$\le n^{-1}|1-\theta|(ET_{0b}P[T_{0b}>n/2]+E(T_{0b}1\{T_{0b}>n/2\}))$$

$$\le 4|1-\theta|n^{-2}ET_{0b}^2 \qquad (1.4)$$

Combining the contributions of (1.2) –(1.3), we thus find tha

$$\Big|\, d_{TV}(L(C[1,b]),L(Z[1,b])) -(n+1)^{-1}\sum_{r\ge 0}P[T_{0b}=r]\left\{\sum_{s\ge 0}P[T_{0b}=s](s-r)(1-\theta)\right\}_+ \Big|$$

$$\le\varepsilon_{\{7.8\}}(n,b)$$

$$=\frac{3}{\theta P_\theta[0,1]}\left\{\varepsilon_{\{10.5(2)\}}(n/2,b)+2n^{-1}ET_{0b}\varepsilon_{\{10.14\}}(n,b)\right\} +2n^{-2}ET_{0b}^2\left\{4+3|1-\theta|+\frac{24|1-\theta|\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\right\} \qquad (1.5)$$

The quantity $\varepsilon_{\{7.8\}}(n,b)$ is seen to be of the order claimed under Conditions $(A_0),(D_1)$ and (B_{12}), provided that $S(\infty)<\infty$; this supplementary condition can be removed if $\phi^*_{\{10.8\}}(n)$ is replaced by $\phi^*_{\{10.11\}}(n)$ in the definition of $\varepsilon_{\{7.8\}}(n,b)$, has the required order without the restriction on the r_i implied by assuming that $S(\infty)<\infty$. Finally, a direct calculation now shows that

$$\sum_{r\ge 0}P[T_{0b}=r]\left\{\sum_{s\ge 0}P[T_{0b}=s](s-r)(1-\theta)\right\}_+ =\frac{1}{2}|1-\theta|E|T_{0b}-ET_{0b}|$$

Example 1.0. Consider the point $O=(0,\dots,0)\in\mathbb{R}^n$. For an arbitrary vector r, the coordinates of the point $x=O+r$ are equal to the respective coordinates of the vector $r: x=(x^1,\dots x^n)$ and $r=(x^1,\dots,x^n)$. The vector r such as in the example is called the position vector or the radius vector of the point x. (Or, in greater detail: r is the radius-vector of x w.r.t an origin O). Points are frequently specified by their radius-vectors. This presupposes the choice of O as the "standard origin". Let us summarize. We have considered $\mathbb{R}^n$ and interpreted its elements in two ways: as points and as vectors. Hence we may say that we leading with the two copies of $\mathbb{R}^n$: $\mathbb{R}^n$ = {points}, $\mathbb{R}^n$ = {vectors}

Operations with vectors: multiplication by a number, addition. Operations with points and vectors: adding a vector to a point (giving a point), subtracting two points (giving a vector). $\mathbb{R}^n$ treated in this way is called an *n-dimensional affine space*. *(*An "abstract" affine space is a pair of sets , the set of points and the set of vectors so that the operations as above are defined axiomatically). Notice that vectors in an affine space are also known as "free vectors". Intuitively, they are not fixed at points and "float freely" in space. From $\mathbb{R}^n$ considered as an affine space we can precede in two opposite directions: $\mathbb{R}^n$ as an Euclidean space $\Leftarrow$ $\mathbb{R}^n$ as an affine space $\Rightarrow$ $\mathbb{R}^n$ as a manifold.Going to the left means introducing some extra structure which will make the geometry richer. Going to the right means forgetting about part of the affine structure; going further in this direction will lead us to the so-called "smooth (or differentiable) manifolds". The theory of differential forms does not require any extra geometry. So our natural direction is to the right. The Euclidean structure, however, is useful for examples and applications. So let us say a few words about it:

Remark 1.0. *Euclidean geometry.* In $\mathbb{R}^n$ considered as an affine space we can already do a good deal of geometry. For example, we can consider lines and planes, and quadric surfaces like an ellipsoid. However, we cannot discuss such things as "lengths", "angles" or "areas" and "volumes". To be able to do so, we have to introduce some more definitions, making $\mathbb{R}^n$ a Euclidean space. Namely, we define the length of a vector $a=(a^1,\dots,a^n)$ to be

$$|a|:=\sqrt{(a^1)^2+\dots+(a^n)^2} \qquad (1)$$

After that we can also define distances between points as follows:

$$d(A,B):=\left|\overrightarrow{AB}\right| \qquad (2)$$

One can check that the distance so defined possesses natural properties that we expect: is it always non-negative and equals zero only for coinciding points; the distance from A to B is the same as that from B to A (symmetry); also, for three points, A, B and C, we have $d(A,B)\le d(A,C)+d(C,B)$ (the "triangle inequality"). To define angles, we first introduce the scalar product of two vectors

$$(a,b):=a^1b^1+\dots+a^nb^n \qquad (3)$$

Thus $|a|=\sqrt{(a,a)}$. The scalar product is also denote by dot: $a.b=(a,b)$, and hence is often referred to as the "dot product" . Now, for nonzero vectors, we define the angle between them by the equality

$$\cos\alpha:=\frac{(a,b)}{|a||b|} \qquad (4)$$

The angle itself is defined up to an integral multiple of 2π . For this definition to be consistent we have to ensure that the r.h.s. of (4) does not exceed 1 by the absolute value. This follows from the inequality

$$(a,b)^2\le|a|^2|b|^2 \qquad (5)$$

known as the Cauchy–Bunyakovsky–Schwarz inequality (various combinations of these three names are applied in different books). One of the ways of proving (5) is to consider the scalar square of the linear combination $a+tb,$ where $t\in R$. As $(a+tb,a+tb)\ge 0$ is a quadratic polynomial in t which is never negative, its discriminant must be less or equal zero. Writing this explicitly yields (5). The triangle inequality for distances also follows from the inequality (5).

Example 1.1. Consider the function $f(x)=x^i$ (the i-th coordinate). The linear function dx^i (the differential of x^i) applied to an arbitrary vector h is simply h^i.From these examples follows that we can rewrite df as

$$df=\frac{\partial f}{\partial x^1}dx^1+\dots+\frac{\partial f}{\partial x^n}dx^n, \qquad (1)$$

which is the standard form. Once again: the partial derivatives in (1) are just the coefficients (depending on x); $dx^1, dx^2, \ldots$ are linear functions giving on an arbitrary vector h its coordinates $h^1, h^2, \ldots,$ respectively. Hence

$$df(x)(h) = \partial_{hf(x)} = \frac{\partial f}{\partial x^1} h^1 + \ldots + \frac{\partial f}{\partial x^n} h^n, \qquad (2)$$

Theorem 1.7. Suppose we have a parametrized curve $t \mapsto x(t)$ passing through $x_0 \in \mathbb{R}^n$ at $t = t_0$ and with the velocity vector $x(t_0) = \upsilon$ Then

$$\frac{df(x(t))}{dt}(t_0) = \partial_\upsilon f(x_0) = df(x_0)(\upsilon) \qquad (1)$$

Proof. Indeed, consider a small increment of the parameter $t : t_0 \mapsto t_0 + \Delta t$, Where $\Delta t \mapsto 0$. On the other hand, we have $f(x_0 + h) - f(x_0) = df(x_0)(h) + \beta(h)|h|$ for an arbitrary vector h, where $\beta(h) \to 0$ when $h \to 0$. Combining it together, for the increment of $f(x(t))$ we obtain

$$f(x(t_0 + \Delta t) - f(x_0)$$
$$= df(x_0)(\upsilon.\Delta t + \alpha(\Delta t)\Delta t)$$
$$+\beta(\upsilon.\Delta t + \alpha(\Delta t)\Delta t).|\upsilon\Delta t + \alpha(\Delta t)\Delta t|$$
$$= df(x_0)(\upsilon).\Delta t + \gamma(\Delta t)\Delta t$$

For a certain $\gamma(\Delta t)$ such that $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$ (we used the linearity of $df(x_0)$). By the definition, this means that the derivative of $f(x(t))$ at $t = t_0$ is exactly $df(x_0)(\upsilon)$. The statement of the theorem can be expressed by a simple formula:

$$\frac{df(x(t))}{dt} = \frac{\partial f}{\partial x^1} x^1 + \ldots + \frac{\partial f}{\partial x^n} x^n \qquad (2)$$

To calculate the value Of df at a point x_0 on a given vector υ one can take an arbitrary curve passing Through x_0 at t_0 with υ as the velocity vector at t_0 and calculate the usual derivative of $f(x(t))$ at $t = t_0$.

Theorem 1.8. For functions $f, g : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$,

$$d(f + g) = df + dg \qquad (1)$$
$$d(fg) = df.g + f.dg \qquad (2)$$

Proof. Consider an arbitrary point x_0 and an arbitrary vector υ stretching from it. Let a curve $x(t)$ be such that $x(t_0) = x_0$ and $x(t_0) = \upsilon$. Hence

$$d(f+g)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t)) + g(x(t)))$$

at $t = t_0$ and

$$d(fg)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t))g(x(t)))$$

at $t = t_0$ Formulae (1) and (2) then immediately follow from the corresponding formulae for the usual derivative Now, almost without change the theory generalizes to functions taking values in $\mathbb{R}^m$ instead of $\mathbb{R}$. The only difference is that now the differential of a map $F : U \to \mathbb{R}^m$ at a point x will be a linear function taking vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$ (instead of $\mathbb{R}$). For an arbitrary vector $h \in| \mathbb{R}^n$,

$$F(x+h) = F(x) + dF(x)(h) + \beta(h)|h| \qquad (3)$$

Where $\beta(h) \to 0$ when $h \to 0$. We have $dF = (dF^1, \ldots, dF^m)$ and

$$dF = \frac{\partial F}{\partial x^1} dx^1 + \ldots + \frac{\partial F}{\partial x^n} dx^n$$

$$= \begin{pmatrix} \frac{\partial F^1}{\partial x^1} \cdots \frac{\partial F^1}{\partial x^n} \\ \ldots \quad \ldots \ldots \\ \frac{\partial F^m}{\partial x^1} \cdots \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \ldots \\ dx^n \end{pmatrix} \qquad (4)$$

In this matrix notation we have to write vectors as vector-columns.

Theorem 1.9. For an arbitrary parametrized curve $x(t)$ in $\mathbb{R}^n$, the differential of a map $F : U \to \mathbb{R}^m$ (where $U \subset \mathbb{R}^n$) maps the velocity vector $x(t)$ to the velocity vector of the curve $F(x(t))$ in $\mathbb{R}^m$:

$$\frac{dF(x(t))}{dt} = dF(x(t))(\dot{x}(t)) \qquad (1)$$

Proof. By the definition of the velocity vector,

$$x(t+\Delta t) = x(t) + \dot{x}(t).\Delta t + \alpha(\Delta t)\Delta t \qquad (2)$$

Where $\alpha(\Delta t) \to 0$ when $\Delta t \to 0$. By the definition of the differential,

$$F(x+h) = F(x) + dF(x)(h) + \beta(h)|h \qquad (3)|$$

Where $\beta(h) \to 0$ when $h \to 0$. we obtain

$$F(x(t+\Delta t)) = F(x + \underbrace{\dot{x}(t).\Delta t + \alpha(\Delta t)\Delta t}_{h})$$

$$= F(x) + dF(x)(\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t) +$$

$$\beta(\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t).\left|\dot{x}(t)\Delta t + \dot{\alpha}(\Delta t)\Delta t\right|$$

$$= F(x) + dF(x)(\dot{x}(t)\Delta t + \gamma(\Delta t)\Delta t$$

For some $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$. This precisely means that $dF(x)\dot{x}(t)$ is the velocity vector of $F(x)$. As every vector attached to a point can be viewed as the velocity vector of some curve passing through this point, this theorem gives a clear geometric picture of dF as a linear map on vectors.

Theorem 1.10 Suppose we have two maps $F:U \to V$ and $G:V \to W$, where $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m, W \subset \mathbb{R}^p$ (open domains). Let $F: x \mapsto y = F(x)$. Then the differential of the composite map $GoF: U \to W$ is the composition of the differentials of F and G:

$$d(GoF)(x) = dG(y)odF(x) \qquad (4)$$

Proof. We can use the description of the differential .Consider a curve $x(t)$ in $\mathbb{R}^n$ with the velocity vector $\dot{x}$. Basically, we need to know to which vector in $\mathbb{R}^p$ it is taken by $d(GoF)$. the curve $(GoF)(x(t) = G(F(x(t))$. By the same theorem, it equals the image under dG of the Anycast Flow vector to the curve $F(x(t))$ in $\mathbb{R}^m$. Applying the theorem once again, we see that the velocity vector to the curve $F(x(t))$ is the image under dF of the vector $\dot{x}(t)$. Hence $d(GoF)(\dot{x}) = dG(dF(\dot{x}))$ for an arbitrary vector $\dot{x}$.

Corollary 1.0. If we denote coordinates in $\mathbb{R}^n$ by $(x^1,\ldots,x^n)$ and in $\mathbb{R}^m$ by $(y^1,\ldots,y^m)$, and write

$$dF = \frac{\partial F}{\partial x^1}dx^1 + \ldots + \frac{\partial F}{\partial x^n}dx^n \qquad (1)$$

$$dG = \frac{\partial G}{\partial y^1}dy^1 + \ldots + \frac{\partial G}{\partial y^n}dy^n, \qquad (2)$$

Then the chain rule can be expressed as follows:

$$d(GoF) = \frac{\partial G}{\partial y^1}dF^1 + \ldots + \frac{\partial G}{\partial y^m}dF^m, \qquad (3)$$

Where dF^i are taken from (1). In other words, to get $d(GoF)$ we have to substitute into (2) the expression for $dy^i = dF^i$ from (3). This can also be expressed by the following matrix formula:

$$d(GoF) = \begin{pmatrix} \frac{\partial G^1}{\partial y^1} \cdots \frac{\partial G^1}{\partial y^m} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial G^p}{\partial y^1} \cdots \frac{\partial G^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial F^1}{\partial x^1} \cdots \frac{\partial F^1}{\partial x^n} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial F^m}{\partial x^1} \cdots \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \ldots \\ dx^n \end{pmatrix} \qquad (4)$$

i.e., if dG and dF are expressed by matrices of partial derivatives, then $d(GoF)$ is expressed by the product of these matrices. This is often written as

$$\begin{pmatrix} \frac{\partial z^1}{\partial x^1} \cdots \frac{\partial z^1}{\partial x^n} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial z^p}{\partial x^1} \cdots \frac{\partial z^p}{\partial x^n} \end{pmatrix} = \begin{pmatrix} \frac{\partial z^1}{\partial y^1} \cdots \frac{\partial z^1}{\partial y^m} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial z^p}{\partial y^1} \cdots \frac{\partial z^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} \cdots \frac{\partial y^1}{\partial x^n} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial y^m}{\partial x^1} \cdots \frac{\partial y^m}{\partial x^n} \end{pmatrix}, \qquad (5)$$

Or

$$\frac{\partial z^\mu}{\partial x^a} = \sum_{i=1}^{m} \frac{\partial z^\mu}{\partial y^i}\frac{\partial y^i}{\partial x^a}, \qquad (6)$$

Where it is assumed that the dependence of $y \in \mathbb{R}^m$ on $x \in \mathbb{R}^n$ is given by the map F, the dependence of $z \in \mathbb{R}^p$ on $y \in \mathbb{R}^m$ is given by the map G, and the dependence of $z \in \mathbb{R}^p$ on $x \in \mathbb{R}^n$ is given by the composition GoF.

Definition 1.6. Consider an open domain $U \subset \mathbb{R}^n$. Consider also another copy of $\mathbb{R}^n$, denoted for distinction $\mathbb{R}^n_y$, with the standard coordinates $(y^1 \ldots y^n)$. A system of coordinates in the open domain U is given by a map $F: V \to U$, where $V \subset \mathbb{R}^n_y$ is an open domain of $\mathbb{R}^n_y$, such that the following three conditions are satisfied :

(1) F is smooth;

(2) F is invertible;

(3) $F^{-1}: U \to V$ is also smooth

The coordinates of a point $x \in U$ in this system are the standard coordinates of $F^{-1}(x) \in \mathbb{R}^n_y$

In other words,

$$F: (y^1 \ldots, y^n) \mapsto x = x(y^1 \ldots, y^n) \qquad (1)$$

Here the variables $(y^1 \ldots, y^n)$ are the "new" coordinates of the point x

Example 1.2. Consider a curve in $\mathbb{R}^2$ specified in polar coordinates as

$$x(t): r = r(t), \varphi = \varphi(t) \qquad (1)$$

We can simply use the chain rule. The map $t \mapsto x(t)$ can be considered as the composition of the maps $t \mapsto (r(t), \varphi(t)), (r, \varphi) \mapsto x(r, \varphi)$. Then, by the chain rule, we have

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \varphi}\frac{d\varphi}{dt} = \frac{\partial x}{\partial r}\dot{r} + \frac{\partial x}{\partial \varphi}\dot{\varphi} \qquad (2)$$

Here $\dot{r}$ and $\dot{\varphi}$ are scalar coefficients depending on t, whence the partial derivatives $\partial x/\partial r, \partial x/\partial \varphi$ are vectors depending on point in $\mathbb{R}^2$. We can compare this with the formula in the "standard" coordinates: $\dot{x} = e_1 \dot{x} + e_2 \dot{y}$. Consider the vectors $\partial x/\partial r, \partial x/\partial \varphi$. Explicitly we have

$$\frac{\partial x}{\partial r} = (\cos\varphi, \sin\varphi) \qquad (3)$$

$$\frac{\partial x}{\partial \varphi} = (-r\sin\varphi, r\cos\varphi) \qquad (4)$$

From where it follows that these vectors make a basis at all points except for the origin (where $r = 0$). It is instructive to sketch a picture, drawing vectors corresponding to a point as starting from that point. Notice that $\partial x/\partial r, \partial x/\partial \varphi$ are, respectively, the velocity vectors for the curves $r \mapsto x(r, \varphi)$ $(\varphi = \varphi_0 \ fixed)$ and $\varphi \mapsto x(r, \varphi)$ $(r = r_0 \ fixed)$. We can conclude that for an arbitrary curve given in polar coordinates the velocity vector will have components $(\dot{r}, \dot{\varphi})$ if as a basis we take $e_r := \partial x/\partial r, e_\varphi := \partial x/\partial \varphi$:

$$\dot{x} = e_r \dot{r} + e_\varphi \dot{\varphi} \qquad (5)$$

A characteristic feature of the basis e_r, e_φ is that it is not "constant" but depends on point. Vectors "stuck to points" when we consider curvilinear coordinates.

Proposition 1.3. The velocity vector has the same appearance in all coordinate systems.

Proof. Follows directly from the chain rule and the transformation law for the basis e_i. In particular, the elements of the basis $e_i = \partial x/\partial x^i$ (originally, a formal notation) can be understood directly as the velocity vectors of the coordinate lines $x^i \mapsto x(x^1, \ldots, x^n)$ (all coordinates but x^i are fixed). Since we now know how to handle velocities in arbitrary coordinates, the best way to treat the differential of a map $F: \mathbb{R}^n \to \mathbb{R}^m$ is by its action on the velocity vectors. By definition, we set

$$dF(x_0): \frac{dx(t)}{dt}(t_0) \mapsto \frac{dF(x(t))}{dt}(t_0) \qquad (1)$$

Now $dF(x_0)$ is a linear map that takes vectors attached to a point $x_0 \in \mathbb{R}^n$ to vectors attached to the point $F(x) \in \mathbb{R}^m$

$$dF = \frac{\partial F}{\partial x^1}dx^1 + \ldots + \frac{\partial F}{\partial x^n}dx^n$$

$$(e_1, \ldots, e_m)\begin{pmatrix} \frac{\partial F^1}{\partial x^1} \cdots \frac{\partial F^1}{\partial x^n} \\ \ldots \ \ldots \ \ldots \\ \frac{\partial F^m}{\partial x^1} \cdots \frac{\partial F^m}{\partial x^n} \end{pmatrix}\begin{pmatrix} dx^1 \\ \ldots \\ dx^n \end{pmatrix}, \qquad (2)$$

In particular, for the differential of a function we always have

$$df = \frac{\partial f}{\partial x^1} dx^1 + ... + \frac{\partial f}{\partial x^n} dx^n, \qquad (3)$$

Where x^i are arbitrary coordinates. The form of the differential does not change when we perform a change of coordinates.

Example 1.3 Consider a 1-form in $\mathbb{R}^2$ given in the standard coordinates:

$A = -ydx + xdy$ In the polar coordinates we will have $x = r\cos\varphi, y = r\sin\varphi$, hence

$$dx = \cos\varphi dr - r\sin\varphi d\varphi$$

$$dy = \sin\varphi dr + r\cos\varphi d\varphi$$

Substituting into A, we get

$$A = -r\sin\varphi(\cos\varphi dr - r\sin\varphi d\varphi)$$

$$+ r\cos\varphi(\sin\varphi dr + r\cos\varphi d\varphi)$$

$$= r^2(\sin^2\varphi + \cos^2\varphi)d\varphi = r^2 d\varphi$$

Hence $A = r^2 d\varphi$ is the formula for A in the polar coordinates. In particular, we see that this is again a 1-form, a linear combination of the differentials of coordinates with functions as coefficients. Secondly, in a more conceptual way, we can define a 1-form in a domain U as a linear function on vectors at every point of U:

$$\omega(\upsilon) = \omega_1\upsilon^1 + ... + \omega_n\upsilon^n, \qquad (1)$$

If $\upsilon = \sum e_i\upsilon^i$, where $e_i = \partial x / \partial x^i$. Recall that the differentials of functions were defined as linear functions on vectors (at every point), and

$$dx^i(e_j) = dx^i\left(\frac{\partial x}{\partial x^j}\right) = \delta^i_j \qquad (2)$$

at every point x.

Theorem 1.9. For arbitrary 1-form ω and path γ, the integral $\int_\gamma \omega$ does not change if we change parametrization of γ provide the orientation remains the same.

Proof: Consider $\left\langle \omega(x(t)), \frac{dx}{dt'} \right\rangle$ and $\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle$ As

$$\left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle = \left| \left\langle \omega(x(t(t'))), \frac{dx}{dt'} \right\rangle \right| . \frac{dt}{dt'},$$

Let p be a rational prime and let $K = \mathbb{Q}(\zeta_p)$. We write ζ for ζ_p or this section. Recall that K has degree $\varphi(p) = p-1$ over $\mathbb{Q}$. We wish to show that $O_K = \mathbb{Z}[\zeta]$. Note that ζ is a root of $x^p - 1$, and thus is an algebraic integer; since O_K is a ring we have that $\mathbb{Z}[\zeta] \subseteq O_K$. We give a proof without assuming unique factorization of ideals. We begin with some norm and trace computations. Let j be an integer. If j is not divisible by p, then ζ^j is a primitive p^{th} root of unity, and thus its conjugates are $\zeta, \zeta^2, ..., \zeta^{p-1}$. Therefore

$$Tr_{K/\mathbb{Q}}(\zeta^j) = \zeta + \zeta^2 + ... + \zeta^{p-1} = \Phi_p(\zeta) - 1 = -1$$

If p does divide j, then $\zeta^j = 1$, so it has only the one conjugate 1, and $Tr_{K/\mathbb{Q}}(\zeta^j) = p-1$ By linearity of the trace, we find that

$$Tr_{K/\mathbb{Q}}(1-\zeta) = Tr_{K/\mathbb{Q}}(1-\zeta^2) = ...$$

$$= Tr_{K/\mathbb{Q}}(1-\zeta^{p-1}) = p$$

We also need to compute the norm of $1-\zeta$. For this, we use the factorization

$$x^{p-1} + x^{p-2} + ... + 1 = \Phi_p(x)$$

$$= (x-\zeta)(x-\zeta^2)...(x-\zeta^{p-1});$$

Plugging in $x = 1$ shows that

$$p = (1-\zeta)(1-\zeta^2)...(1-\zeta^{p-1})$$

Since the $(1-\zeta^j)$ are the conjugates of $(1-\zeta)$, this shows that $N_{K/\mathbb{Q}}(1-\zeta) = p$ The key result for determining the ring of integers O_K is the following.

LEMMA 1.9

$$(1-\zeta)O_K \cap \mathbb{Z} = p\mathbb{Z}$$

Proof. We saw above that p is a multiple of $(1-\zeta)$ in O_K, so the inclusion $(1-\zeta)O_K \cap \mathbb{Z} \supseteq p\mathbb{Z}$ is immediate. Suppose now that the inclusion is strict. Since $(1-\zeta)O_K \cap \mathbb{Z}$ is an ideal of $\mathbb{Z}$ containing $p\mathbb{Z}$ and $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, we must have $(1-\zeta)O_K \cap \mathbb{Z} = \mathbb{Z}$ Thus we can write

$$1 = \alpha(1-\zeta)$$

For some $\alpha \in O_K$. That is, $1-\zeta$ is a unit in O_K.

COROLLARY 1.1 For any $\alpha \in O_K$, $Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) \in p\mathbb{Z}$

PROOF. We have

$$\begin{aligned} Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) &= \sigma_1((1-\zeta)\alpha)+...+\sigma_{p-1}((1-\zeta)\alpha) \\ &= \sigma_1(1-\zeta)\sigma_1(\alpha)+...+\sigma_{p-1}(1-\zeta)\sigma_{p-1}(\alpha) \\ &= (1-\zeta)\sigma_1(\alpha)+...+(1-\zeta^{p-1})\sigma_{p-1}(\alpha) \end{aligned}$$

Where the σ_i are the complex embeddings of K (which we are really viewing as automorphisms of K) with the usual ordering. Furthermore, $1-\zeta^j$ is a multiple of $1-\zeta$ in O_K for every $j \neq 0$. Thus $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in (1-\zeta)O_K$ Since the trace is also a rational integer.

PROPOSITION 1.4 Let p be a prime number and let $K = \mathbb{Q}(\zeta_p)$ be the p^{th} cyclotomic field. Then $O_K = \mathbb{Z}[\zeta_p] \cong \mathbb{Z}[x]/(\Phi_p(x))$; Thus $1, \zeta_p, ..., \zeta_p^{p-2}$ is an integral basis for O_K.

PROOF. Let $\alpha \in O_K$ and write $\alpha = a_0 + a_1\zeta + ... + a_{p-2}\zeta^{p-2}$ With $a_i \in \mathbb{Z}$. Then

$$\alpha(1-\zeta) = a_0(1-\zeta) + a_1(\zeta - \zeta^2) + ... + a_{p-2}(\zeta^{p-2} - \zeta^{p-1})$$

By the linearity of the trace and our above calculations we find that $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) = pa_0$ We also have $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in p\mathbb{Z}$, so $a_0 \in \mathbb{Z}$ Next consider the algebraic integer $(\alpha - a_0)\zeta^{-1} = a_1 + a_2\zeta + ... + a_{p-2}\zeta^{p-3}$; This is an algebraic integer since $\zeta^{-1} = \zeta^{p-1}$ is. The same argument as above shows that $a_1 \in \mathbb{Z}$, and continuing in this way we find that all of the a_i are in $\mathbb{Z}$. This completes the proof.

Example 1.4 Let $K = \mathbb{Q}$, then the local ring $\mathbb{Z}_{(p)}$ is simply the subring of $\mathbb{Q}$ of rational numbers with denominator relatively prime to p. Note that this ring $\mathbb{Z}_{(p)}$ is not the ring $\mathbb{Z}_p$ of p-adic integers; to get $\mathbb{Z}_p$ one must complete $\mathbb{Z}_{(p)}$. The usefulness of $O_{K,p}$ comes from the fact that it has a particularly simple ideal structure. Let a be any proper ideal of $O_{K,p}$ and consider the ideal $a \cap O_K$ of O_K. We claim that $a = (a \cap O_K)O_{K,p}$; That is, that a is generated by the elements of a in $a \cap O_K$. It is clear from the definition of an ideal that $a \supseteq (a \cap O_K)O_{K,p}$. To prove the other inclusion, let α be any element of a. Then we can write $\alpha = \beta/\gamma$ where $\beta \in O_K$ and $\gamma \notin p$. In particular, $\beta \in a$ (since $\beta/\gamma \in a$ and a is an ideal), so $\beta \in O_K$ and $\gamma \notin p$. so $\beta \in a \cap O_K$. Since $1/\gamma \in O_{K,p}$, this implies that $\alpha = \beta/\gamma \in (a \cap O_K)O_{K,p}$, as claimed.We can use this fact to determine all of the ideals of $O_{K,p}$. Let a be any ideal of $O_{K,p}$ and consider the ideal factorization of $a \cap O_K$ in O_K. write it as $a \cap O_K = p^n b$ For some n and some ideal b, relatively prime to p. we claim first that $bO_{K,p} = O_{K,p}$. We now find that

$$a = (a \cap O_K)O_{K,p} = p^n bO_{K,p} = p^n O_{K,p}$$

Since $bO_{K,p}$. Thus every ideal of $O_{K,p}$ has the form $p^n O_{K,p}$ for some n; it follows immediately that $O_{K,p}$ is noetherian. It is also now clear that $p^n O_{K,p}$ is the unique non-zero prime ideal in $O_{K,p}$. Furthermore, the inclusion $O_K \mapsto O_{K,p}/pO_{K,p}$ Since $pO_{K,p} \cap O_K = p$, this map is also surjection, since the residue class of $\alpha/\beta \in O_{K,p}$ (with $\alpha \in O_K$ and $\beta \notin p$) is the image of $\alpha\beta^{-1}$ in $O_{K/p}$, which makes sense since β is invertible in $O_{K/p}$. Thus the map is an isomorphism. In particular, it is now abundantly clear that every non-zero prime ideal of $O_{K,p}$ is maximal. To show that $O_{K,p}$ is a Dedekind domain, it remains to show that it is integrally closed in K. So let $\gamma \in K$ be a root of a polynomial with coefficients in $O_{K,p}$; write this polynomial as

$$x^m + \frac{\alpha_{m-1}}{\beta_{m-1}}x^{m-1} + ... + \frac{\alpha_0}{\beta_0}$$ With $\alpha_i \in O_K$ and $\beta_i \in O_{K-p}$. Set $\beta = \beta_0\beta_1...\beta_{m-1}$. Multiplying by β^m we find that $\beta\gamma$ is the root of a monic

polynomial with coefficients in O_K. Thus $\beta\gamma \in O_K$; since $\beta \notin p$, we have $\beta\gamma / \beta = \gamma \in O_{K,p}$. Thus $O_{K,p}$ is integrally close in K.

COROLLARY 1.2. Let K be a number field of degree n and let α be in O_K then

$$N'_{K/□}(\alpha O_K) = \left|N_{K/□}(\alpha)\right|$$

PROOF. We assume a bit more Galois theory than usual for this proof. Assume first that $K/□$ is Galois. Let σ be an element of $Gal(K/□)$. It is clear that $\sigma(O_K)/\sigma(\alpha) \cong O_{K/\alpha}$; since $\sigma(O_K) = O_K$, this shows that $N'_{K/□}(\sigma(\alpha)O_K) = N'_{K/□}(\alpha O_K)$. Taking the product over all $\sigma \in Gal(K/□)$, we have $N'_{K/□}(N_{K/□}(\alpha)O_K) = N'_{K/□}(\alpha O_K)^n$ Since $N_{K/□}(\alpha)$ is a rational integer and O_K is a free□ -module of rank n,

$O_K / N_{K/□}(\alpha)O_K$ Will have order $N_{K/□}(\alpha)^n$; therefore

$$N'_{K/□}(N_{K/□}(\alpha)O_K) = N_{K/□}(\alpha O_K)^n$$

This completes the proof. In the general case, let L be the Galois closure of K and set $[L:K] = m$.

In this paper we have presented the fabrication of a monolithically integrated tunable laser source in the 1700 nm wavelength region. This laser is meant to be used for optical coherence tomography in medical applications. Important requirements on this laser for use in medical applications are as follows:

1) An output wavelength range around 1700 nm to reduce absorption due to water in the human tissue and a reduction in Rayleigh scattering;
2) tuning bandwidth of ideally more than 100 nm which mostly defines the image depth resolution;
3) Laser linewidth less than 0.07 nm to get a coherence length of at least 6 mm necessary for the imaging depth;
4) scan rate of at least 20 kHz for patient comfort and to reduce imaging artifacts caused by patient movement; and
5) output power of 1 mW. Other advantages of a monolithically integrated laser source are the miniaturization compared to the current expensive bulky laser systems. This miniaturization can reduce the cost and the power consumption of the system. Furthermore, due to the use of voltage controlled electro-optically PHMs the calibration of the filters remains stable over a long period as discussed in [17]. The arbitrary sequence in which the filters can be tuned makes it possible to scan the laser also linear in the frequency domain which is preferable in real time OCT measurements due to the reduced calculation time in Fourier transformations on data with a linear frequency scale. If we compare the laser presented above with these requirements we can state that we made a large step towards the realization of the desired tunable laser. We were able to realize a monolithically integrated tunable laser source in the 1700 nm wavelength region due to the integration of special designed QD-amplifiers into the active-passive integration technology. The fabrication of such a relatively large and complex chip is more sensitive to fatal defects in the wafer and the processing due to the large chip size. The performance of the laser presented in this work approaches or satisfies most of the required performance specifications, while others still need some improvements to reach these requirements. We will summarize the results, compare them with the requirements and discuss if and how these requirements can be obtained.

A. Tuning Bandwidth

The measured tuning bandwidth of the laser system was 60 nm where more than 100 nm is ideal. This results in a maximum depth resolution of 22 μm (in vacuum). The limitation on the measured bandwidth is imposed by the limitation of the roundtrip gain in the ring laser cavity. This roundtrip gain was limited by the fact that laser starts switching directions at higher current levels due to the fact that the provision for making the ring unidirectional did not function. The tuning range of the filters is far wider than the tuning bandwidth of the laser and did not limit the laser tuning bandwidth. Increasing the roundtrip gain, especially at the edge of spectrum, will directly improve the tuning bandwidth. This can be done in several ways; reducing the passive length of the ring laser cavity, reducing the waveguide losses in the passive waveguides, increasing the length of the gain sections or the gain per unit length and flattening of the gain spectrum. The first three solutions are a matter of optimization of the design. For example the passive length can be reduced by rotating the HR-filter 180 degrees which reduces the cavity by approximately 6 mm. This will introduce other problems, such as the location of the bond pads and the polyimide planarization in the PHM regions as discussed in [17]. The passive waveguide losses can be reduced by optimizing the layerstack for 1700 nm wavelength, however this will make the required fabrication technology move away from the generic integration technology. Increasing the gain per unit length in these QD-amplifiers is less straightforward. This increase in gain has however been demonstrated in a QD layerstack by growing quantum-dots on quantum wells which increases the quantum-dot density [26] and consequently the modal gain. Flattening of the gain spectrum in these QDamplifiers can in principle be done during the growth process by introducing a chirped central wavelength over different QD layers as has been demonstrated in the InGaAs-QD system [27]. This however also reduces the maximum gain, which can again reduce the tuning bandwidth. A broad gain spectrum of 140 nm at 1.6 μm wavelength has also been demonstrated with InAs/InGaAlAs quantum-dash-in-well amplifiers on InP substrate [28] however it is unknown if this can be extended to the 1.7 μm wavelength region. Another possibility which would require additional research, is the use of strained Quantum Wells (QW) instead of quantum-dots. The advantage of QW above QD is the larger gain per unit length, however the gain bandwidth is normally much more limited to approximately 40 nm. The gain bandwidth can in principle be increased by designing each QW to have a different peak wavelength, also called chirped QWs [29].

B. Laser Linewidth

The effective linewidth of the laser is measured to be 0.1 nm

where 0.07 nm was required to get a coherence length >6 mm.

From Fig. 9d we could see that the FWHM of the laser peak can be less than the 0.07 nm and is often 0.05 nm or less. The effective linewidth is however limited by what looks like mode hopping and dual mode operation at the 0.1 nm mode spacing observed in the laser. This mode pattern causes a broadening of the laser FWHM while tuning in between two modes. The origin of this 0.1 nm cavity mode spacing has not yet been identified. A possible explanation of the 0.1 nm spacing can be the combination of the 0.02 nm longitudinal mode spacing of the total ring cavity together with the 0.05 nm etalon transmission peak spacing caused by reflections from the amplifier ends.

C. Scan Rate

The maximum scan rate is determined by the switching time between two wavelength settings and the number of wavelength steps over the tuning range of the laser. The switching time between two wavelengths has been measured to be 500 ns. Assuming the ideal 0.05 nm wavelength steps are used this implies 600 μs for a 60 nm scan. This 1.67 kHz scan rate is just over one order of magnitude less than the desired 20 kHz scan rate. To increase the scan speed we have to focus on the reduction of the switching time. First of all the step to step switching of the wavelength during a scan is probably faster. This is due to the lower suppression at 0.05 nm from the starting wavelength than at 45 nm as has been done during the measurement. The neighboring cavity modes will already be at a higher power level than cavity modes much further away. The switching time is mainly determined by the roundtrip gain, necessary to build up the laser peak. Increasing the roundtrip gain will reduce the switching time. Options to increase the gain have been presented above. We have to take into account that the non-uniform gain over the tuning bandwidth also introduces a non-uniform maximum tuning speed over the tuning bandwidth. In most tunable lasers it is not possible to utilize this fact due to the fixed linear or sinusoidal wavelength tuning mechanism. However with the laser presented each wavelength step is individually controlled. This means that the scan speed can be changed over the tuning range. Furthermore, the suppression of the clock wise direction will also increase the small signal gain for the mode building up and so reduce the switching time. A decrease in the ring cavity length reduces the roundtrip time and therefore the switching time.

D. Output Power

The measured 0.05–0.15 mW output power is one order of magnitude lower than the preferred 1 mW output power. The output power can be increased by (approximately a factor two) suppressing the clock-wise direction. The rest should be done by increasing the roundtrip gain.

E. Overall Conclusion

A monolithically integrated continuously tunable laser for OCT fulfilling all requirements stated above can be realized in the InP-based active-passive integration technology combined with QD active areas. The tunable laser system presented in this work does not fulfill all requirements but neither do its limitations appear unsolvable. Most improvements can be made by increasing the roundtrip gain [26], making the ring unidirectional and solving the issue of the 0.1 nm interference pattern. We presented a methodology to control the balloon inflation inside deformable structures such as arteries, based on feedback from IVOCT imaging. Using this methodology, we successfully orchestrated the movements of a syringe plunger with the acquired real-time IVOCT images to control luminal diameter. To our knowledge, this was the first demonstration of a controlled clinical application, where IVOCT images were processed in real time in a feedback loop. The combination of IVOCT and control engineering results in a technology that could benefit medical device industry, researchers, and clinical users. For industry, it could provide predetermined and consistent conditions for testing of angioplasty devices. For researchers, it could provide precisely controlled conditions to validate the results of mechanical tests as well as mathematical models, these tests and models being used to improve the design of balloons and stents. For clinical users, a controlled inflation could be a very relevant tool to insure safe percutaneous coronary intervention procedures. The main technical challenge in bringing this technology to the clinical world is the integration of the OCT probe in commercial percutaneous coronary intervention devices. This is an engineering challenge that can be overcome with existing technology. In future study, more complicated control algorithms, e.g., PID controllers, should be applied to improve the performance characteristics. The control algorithm should be designed to also incorporate constraints on the inflation pressures. The technology should also be validated *in vivo* by performing controlled inflations for angioplasty and stent deployment in animal models.

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